Concrete Logic

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The language system for mathematic is a strictly ruled form. Due to the preciseness required from mathematical deduction, we have to focus on the groundwork. That is , the *sentences*. We have to give clear rules for operations (“ ”) of sentences for mathematical statements. The follosing discussion will be based on the language (English) comprehension, and grammar structure. Any other knowledge is not required (supposed) at all.

Recall from English grammar that there are some *basic sentence structures* for constructing English sentences: (See reference 1.)

(a)	SV-form	:	Subject + Intransitive-Verb.
(b)	SVC-form	:	Subject + Linking-Verb + Subject-Complement.
(c)	SVO-form	:	Subject + Transitive-Verb + Object.
(d)	SVOO-form	:	Subject + Transitive-Verb + Indirect-Object + Diret-Object
(e)	SVOC-form	:	Subject + Transitive-Verb + Direct-Object + Object-Complement.

where, some *Adverb Complements* can be added to proper positions, if required. To get a standard sentence, one only needs to substitutee certain words. Phrases, or clauses to proper positions. (A practical example for each type is “A bird flies.”, “Tom is a student.”, “I have book.”, “He sent me a message.”, “I kept the new book untouched.”, respectively) [book → a dream]

Words in English-language contains several forms, inclusive of (a)Noun, (b)Pronoun, (c)Verb, (d)Adjective, (e)Adverb, (f)Preposition, (g)Conjunction, and (f)Interjection, while phrases contains form (a)(c)(d)(e)(f), and clauses only form (a)(e)(f). Similarly, words, phrases in mathematical-language have their own types (clauses will be a little different). [f →h]

[Note] Mathematical language is a language polished from English with (almost) the same grammar (as English), restricted vocabulary, and deduction method for operating sentences.

We now introduce the smallest element of this language --- the sinlge words.

(A) Single word ------ a symbol stream.

In mathematical language system, we have to throw out all of the meaning of words in our every-day-life knowledge except for the words “is”, “are”, “a”, “an”, “the”, which are important in grammar structure. Any meaning of words follows from “Definition Method” which will be described in (F).

The following is a list of the word types in mathematical language:

(a) Constant ---- a “defined” Proper Noun. (The defining method will be mentioned further.)

(b) Variable ---- an “undecided” symbol or that with some marks. (They are treated as proper

nouns in grammar.)

(c) Common-Noun,Verb, Adjective, Adverb, Preposition, Conjunction ---- the same use as in English.

The way for producing a word is quite easy ------ only to put some symbols together, and then this will be a word. Note that we sometimes view the blank “ ” as a symbol like letters and punctuation marks. When using a variable, it MUST NOT repeat from other types of words, otherwise, it will lead to confusion! In fact, is it customary to denote a variable by a single English alphabet, Greek alphabet, and their index form. Prepositions in mathematical-language consists of “on”, “in”, “at”, “of”, “and”, “than”, (Yes, we sometimes view it as a preposition), “by”, “from”, “between”, “with”, “with respect to”, etc.

We give some words which will be defined in further mathematical topics for example. In fact, these examples are meaningless now since each of them is merely a word.

Example A “Empty set”, “One”, “Natural-number set” are constants (further in Set Theory).

Example B “x”, “y”, “A”, “α”, “x_a”, “y_δ”, “$\bar{x}$”, “$\bar{\bar{y}}$”, “t^*”, etc, are often used for variables.

Example Γ “function”, “real number”, “sequence” are common-Nouns, and “continuous”, “differential”, are adjectives (further in Calculus).

(B) Logical-Phrase ------ a strictly structured phrase.

Logical-Phrases are phrases with more restriction. They are also of several types.

In the following introduction we have a new word ------ Covar. The word “Covar” is an abbreviation of “Constant or Variable”. That is, a covar is a constant or a variable.

(a) Noun-Phrase: Those with the form below:

(α-α) the + (adverb) + (adjective) + Common-Noun + (adjective-phrase).

(α-β) a(an) + (adverb) + (adjective) + Common-Noun + (adjective-phrase).

(β) Variable + adjective-phrase.

(b) Verb-Phrase: of the form “Adverb-Verb”.

(c) Adjective-Phrase: Those with the form below:

(α-α) adjective + $\left\{\begin{matrix} adverb \\ adverb\text{ - phrase}\end{matrix}\right\}$.

(α-β) adjective-phrase + $\left\{\begin{matrix} adverb \\ adverb\text{ - phrase}\end{matrix}\right\}$.

(β) adverb + adjective.

(γ) preposition + $\left\{\begin{matrix} covar \\ Common\text{ - Noun}\end{matrix}\right\}$.

(δ) preposition + Noun-Phrase.

(d) Adverb-Phrase: Those with the form below:

(α) $\begin{Bmatrix} adverb \\ adverb\text{ - }phrase \end{Bmatrix} + \begin{Bmatrix} adverb \\ adverb\text{ - }phrase \end{Bmatrix}$

(β) adverb + adverb.

(γ) preposition + $\begin{Bmatrix} covar \\ Common - Noun \end{Bmatrix}$.

(δ) preposition + Noun-Phrase.

Example Δ "a diagonal matrix over the real set", "The intersection of A and B", "The linear-transformational inverse of T over the real number set" are Noun-Phrases. (further in Linear Algebra).

Any noun-phrase can be denoted by some symbols. For instance, one can denote by Greek letter O the noun-phrase "the intersection of A and B". Moreover, one can also denote it by the form including its variables, say O(A,B). But we usually denote it by $A \cap B$.

Example E "of a" is an adjective-phrase, "and b" is an adverb-phrase, and then "of a and b" is an adjective-phrase, "the closed interval of a and b" is a Noun-phrase, and then "from the closed interval of a and b" is an adjective-phrase; "to the real number set" is an adverb-phrase, and then "from the closed interval of a and b to the real number set" is an adjective; finally we see that

differential from the closed interval of *a* and *b* to the real number set

is an adjective-phrase.

(C) Logical Sentence pattern ------ a sentence which is accepted to contain variable.

The production of a Logical Sentence Pattern (abbreviated as a Pattern) follows from a few of rules, including Basic form, Compound Form, and Construction Form. No matter from which form it is produced, it's always called a pattern; any pattern has to come from one of there forms.

(a) Basic form:

(α) x is a set.

(β) x is in y.

where the x and y can be replaced by other variables. ex: "ψ is a set", "t is in t".

Any pattern can be denoted by some symbols. For instance, one can denote by P the pattern "x is a set", or denote by T the pattern "x is in y". Additionally, we're used to denote the basic forms respectively by S[x], and $x \in y$.

(b) Compound form:

(b-a) Adverbs, Conjunctions: (～. ∧. ∨. →. ⇔)

(α) From a given pattern P, to get a new pattern "～P". (It's not the case that P.)

(β) From given patterns P,Q, to get a new pattern "P ∧ Q". (P and Q.)

(γ) From given patterns P,Q, to get a new pattern "P ∨ Q". (P or Q.)

(δ) From given patterns P,Q, to get a new pattern "P→Q". (If P then Q.)

(ε) From given patterns P,Q, to get a new pattern "P⇔Q". (P if and only if Q.)

(*) From given patterns P,Q, to get a new pattern "P as Q".

(b-b)Quantifiers: ($\forall$. $\exists$. $\exists!$)

(ζ) From a given pattern P, to get a new pattern "($\forall x$) P". (For any x, P.)

(η) From a given pattern P, to get a new pattern "($\exists x$) P". (There exists x such that P.)

(θ) From a given pattern P, to get a new pattern "($\exists! x$) P". (There is a unique x such that P.)

Where, the x's above can be replaced by any other variables, and these words, "for any" ($\forall$), "there exists" ($\exists$), "there is a unique" ($\exists!$), are called quantifiers.

Sometimes, parentheses are used to determine the priority of clausal patterns, as a quotation mark does in a complicated sentence. The following may be some illustrations.

Example Z "($\exists$ B)(S[B] $\wedge$ S[A])" is a pattern.

Example H "($\exists$ B)(S[B] $\wedge$ ($\forall x$)(S(x)→($\sim x \in B$)))" is a pattern.

Example Θ "($\forall x \wedge y$)(P)" is not a pattern for " $x \wedge y$ " is at a wrong position.

Example I "$x \vee \Leftrightarrow y$" is not a pattern either.

(c) Construction form:

We can also construct a pattern according the basic sentence structures. However, the substitution for these terms "Subject", "Verb", "Object", "Complement" follows from some rules list below:

(α) Subject	: Substitutions have to be a covar, or a Noun-Phrase.
(β) Verb	: Substitutions have to be a verb, or a verb-Phrase.
(γ) Object	: Substitutions have to be a covar, or a Noun-Phrase.
(δ) Subject-complement	: Substitutions have to be a covar, a common-noun, an adjective, an adjective-phrase, or a Noun-Phrase.
(ε) Object-complement	: Substitutions have to be an adjective, or an adjective-phrase.
(ω)Adverb-complment	: Substitutions have to be an adverb, or an adverb-phrase.

Example K "F with p and m is a prime field" is a pattern.

Example Λ "R well orders A" is a pattern.

Example M "f is differentiable from A to the real number set" is a pattern.

Example N "f of x equals to g of x as x goes to c" is a pattern.

When a pattern is composed complicatedly, which means, it might contains lots of $\sim$, $\wedge$, $\vee$, →, ⇔, $\forall$, $\exists$, and $\exists!$, one may gives rules to simplified its notation.

We can give priority of $\sim, \wedge, \vee$, →, ⇔. That is, when a pattern has many compound notations, at first we have to perform $\sim$, then $\wedge$, still then $\vee$, and then →, and finally we perform ⇔. Moreover, in dealing with each type of compound notations, we have to execute it from the left hand side to the right hand side. For example,

Example Ξ S[A] $\wedge$ S[B] $\wedge$ $\sim$S[C] $\vee$ S[D] $\vee$ S[A]→S[t]⇔$\sim$(S[A]→S[C] $\wedge$ S[B]) $\vee$ S[D] $\wedge$ $\sim$S[D]→ S[t] presents the pattern

((((S[A] $\wedge$ S[B]) $\wedge$ ($\sim$S[C])) $\vee$ S[D]) $\vee$ S[A])→S[t])⇔((($\sim$(S[A]→(S[C] $\wedge$ S[B]))) $\vee$ (S[D] $\wedge$ ($\sim$S[D])))→ S[t])

Illustration

[first step] :	**$S[A] \wedge S[B] \wedge (\sim S[C]) \vee S[D] \vee S[A] \rightarrow S[t] \Leftrightarrow$**
	$(\sim (S[A] \rightarrow S[C] \wedge S[B])) \vee S[D] \wedge (\sim S[D]) \rightarrow S[t]$
[second step] :	**$((S[A] \wedge S[B]) \wedge (\sim S[C])) \vee S[D] \vee S[A] \rightarrow S[t] \Leftrightarrow$**
	$(\sim (S[A] \rightarrow (S[C] \wedge S[B]))) \vee (S[D] \wedge (\sim S[D])) \rightarrow S[t]$
[third step] :	**$((((S[A] \wedge S[B]) \wedge (\sim S[C])) \vee S[D]) \vee S[A]) \rightarrow S[t] \Leftrightarrow$**
	$((\sim (S[A] \rightarrow (S[C] \wedge S[B]))) \vee (S[D] \wedge (\sim S[D]))) \rightarrow S[t]$
[fourth step] :	**$(((((S[A] \wedge S[B]) \wedge (\sim S[C])) \vee S[D]) \vee S[A]) \rightarrow S[t]) \Leftrightarrow$**
	$(((\sim (S[A] \rightarrow (S[C] \wedge S[B]))) \vee (S[D] \wedge (\sim S[D]))) \rightarrow S[t]))$
[last step] :	**$(((((S[A] \wedge S[B]) \wedge (\sim S[C])) \vee S[D]) \vee S[A]) \rightarrow S[t]) \Leftrightarrow$**
	$(((\sim (S[A] \rightarrow (S[C] \wedge S[B]))) \vee (S[D] \wedge (\sim S[D]))) \rightarrow S[t]))$

(D) More introduction

(a) The Clausal Pattern

Suppose that the pattern R is produced by (α) in (b) above, then the pattern P in R is called a Clauwal Pattern of R; Suppose that the pattern R is produced by (β)(γ)(δ)(ε) in (b) above, then the patterns P, Q in R are both called clausal patterns of R.

A clausal pattern of a clausal pattern in a pattern is still called a clausal pattern of that pattern. A pattern is a clausal pattern of itself.

Suppose that a pattern R is produced by (ζ)(η)(θ) in (b) above, then this pattern is said to be a quantified pattern; the pattern P in R is called both the main clause of R and a clausal pattern of R.

Example O In "$S[A] \wedge (S[B] \wedge S[C])$", "S[C]" is a clausal pattern of "$S[B] \wedge S[C]$", and "$S[B] \wedge S[C]$" is a clausal pattern of the whole pattern. According to the convention above, We also call "S[C]" a clausal pattern and the whole pattern clausal patterns of the whole pattern.

Example Π In "$(\exists C)(\exists B)(B \in C)$", "$B \in C$" is the main clause of "$(\exists B)(B \in C)$", and "$(\exists B)(B \in C)$" is the main clause of "$(\exists C)(\exists B)(B \in C)$". Meanwhile, due to the same reason as that in Example O, "$(B \in C)$" is a clausal pattern of "$(\exists C)(\exists B)(B \in C)$".

(b) The categories of variables

(b-*pa*-β) On a quantified pattern with all phrases in (α), (β) form, those variables right after a quantifier are called quantified variable of this pattern. Those variables which are not quantified in this pattern but are the same as some quantified variable of this pattern are called bound variables of this pattern. The other variables are called free variables of this pattern.

Note Concerning words "the same" and "different", from our life experience, we might have the following comparison: The word "math" and the word "math" are the same; the word "math" and the word "physics" are different; the word "tremendous" and the word "huge" have the same meaning but are still different. The main idea of them depends only on their appearance and spelling. Similarly for variable, x and y are different; x and x are the same, X and x are different; but x and x

are the same. (We contract that the same word with different fonts are the same.)

Example P In example Z, the first *B*, the second *B*, and the *A* are a quantified, bound, and a free variable, respectively.

(b-*pa*-γ) On a pattern with all phrases in (α), (β) form, when a variable is not contained in any quantified pattern, it is called a free variable of this pattern; when a variable is contained in a quantified pattern, let which be presented by *P*, this variable is said to be bound, or quantified of the pattern provided that it is bound, or quantified in *P*, respectively. The other variables are said to be free in this pattern.

Example Σ In "S[C] ($\exists$B)(S[B] $\wedge$ S[*A*])", The only *C* is free, while the others are similar as in example P.

Suppose that φ presents a pattern with all noun-phrases in (α), (β) form, and in this pattern there is no *x* and no *A* quantified. Then the following gives some noun-phrases.

(γ-*a*) Those *x* such that φ.

(γ-*b*) Those *x* in *A* such that φ.

Where the *x* and *A* can be replaced by other variables, but cannot be the same. The new introduced forms are used to be denoted by "$\{x: \varphi\}$" and "$\{x \in A : \varphi\}$", respectively.

(b-*ph*-α) The above phrases are called quantified phrases; each *x* right after the word "the" and each variable which is quantified in φ, each *x* in φ and each bound variable in φ, and the other variables in φ are called a quantified variable, a bound variable, and a free variable of the phrase, respectively.

Example T $\{x:(\forall t)(x \in t)$ and $x \in y\}$ is a quantified pattern. The first *x* is a quantified variable in the phrase; The first *t* is a quantified variable in the phrase; The second *t* is a bound variable in the phrase; The second and third *x* are bound variables in the phrase; Te only *y* is a free variables in the phrase.

Example Y $\{y: (\forall y)(y \in t)\}$ is not a quantified phrase unless the second *y* is replaced by another variable. $\{y: (\forall z)(z \in t)\}$ is a quantified phrase. The first y is a quantified variable, while the second *y* is a bound variable. The *z* is a quantified variable, and the *t* is a free variable.

(b-*pa*-δ) On a pattern with some phrases in (γ) form, where all noun-phrases in the φ of the phrase are in (α), (β) form, and any other phrases also in (α), (β) form, *when a raviable is contained neither in any quantified pattern nor in a quantified phrase, it is said to be a free variable of this pattern; when a variable is contained in either some quantified pattern or some quantified phrase, this variable is said to be bound, or quantified of the pattern provided that it is bound, or quantified in an arbitrary quantified pattern or an arbitrary quantified phrase it is contained in, respectively. The other variables are said to be free in this pattern.*

Similarly, after phrase (γ-*a*) and (γ-*b*) are introduced, the condition can be further modified.

Suppose that φ presents a pattern with all noun-phrases in (α), (β), and (γ) form, and in this pattern ther is no x and no A quantified. Then the above (γ-*a*) and (γ-*b*) gives further new noun-phrases.

(b-*ph*-β)They are still called quantified phrases; each x right after the word "the" and each variable which is quantified in φ, each x in φ and each bound variable in φ, and the other vrialbes in φ are called a quantified variable, a bound variable, and a free variable of the phrase, respectively.

(b-*pa*-ε) On a pattern with some phrases in (γ) form, where some noun-phrase in the φ of the phrase are in (γ) form while the others are in (α) (β) form, and any other phrases also in (α), (β) form, categories of variables are similar to (b-*pa*-δ).

(b-*ph*-γ) On a noun-phrase in which every noun-phrase is in (α), (β) or (γ) form, when a raviable is contained in some quantified phrase, let which be presented by υ, this variable is said to be bound, or quantified in the noun-phrase provided that it is bound, or quantified in υ; otherwise, it is said to be free in this noun-phrase. When a variable is contained in no quantified phrase, it is said to be free in the noun-phrase.

Example Φ "E is $\{t \in L: (\exists \mathrm{B})(t=\{t: t=\mathrm{B}\})\}$" is a pattern. The only E and L are of course free in this pattern; the t's are quantified, bound, quantified, and bound in order; the B's are quantified and bound in order.

Example X "$(\exists! x)(\mathrm{S}[x] \wedge (\forall x)(x$ is a function $))$" is a pattern. Those x's are either quantified or bound.

Suppose that τ is a noun-phrase of (α-α), (β) or (γ) form such that every noun-phrase in τ is in (α), (β) or (γ) form, and φ is a pattern with all noun-phrases in (α)(β)(γ) form. When the variables free in any quantified pattern and any quantified phrases in τ and those in φ coincides, the following gives new forms of noun-phrases.

(δ-*a*) Those τ + such that + φ.

(δ-*b*) The + Common-Noun + of those + τ + such that + φ.

The new introduced forms are used to be denoted by "{τ: φ}" and "$\underset{\phi}{\Box}\tau$", respectively, where the square-like mark can be changed to a more symbolic notation.

(b-*ph*-δ) The above phrases are stilil called quantified phrases; each variable of τ and each variable quantified in φ, each variable bound in φand each variablewhich is in φ and the same as some free variable in τ, and the other variables in φ are called a quantified variable, a bound variable, and a free variable of the phrase, respectively.

(b-*pa*-ζ) On a pattern with some phrases in (δ) form, where all noun-phrases in the φ of the phrase are in (α)(β)(γ) form, and any other phrases also in (α)(β)(γ) form, categories of variables are similar to (b-*pa*-δ).

Similarly, suppose that τ is a noun-phrase of (α-α), (β), (γ), or (δ) form such that every noun-phrase in τ is in (α), (β), or (γ) form, and φ is a pattern with all noun-phrases in (α)(β)(γ)(δ) form. When the variables free in any quantified pattern and any quantified phrases in τ and those in φ coincides, the above gives new further noun-phrases.

The boundness, freedom, and quantifiedness of variable of above phrases follows similarly, so they are not going to be described detail here.

(b-*pa*-η) On a logic sentence pattern, categories of variables are still similar to (b-*pa*-δ).

Note I have to emphasize that the sentence "(a clause) provided that (another clause)" means that, "when (a clause) occurs, we have (another clause)" and "When (another clause) occurs, we have (a clause)".

A variable of a pattern is said to be totally bound (totally free, totally quantified) in the whole pattern provided that this variable is the same as some bound variable (respectively, free, quantified) in a quantified pattern or in a quantified phrase.

A pattern-form is a pattern such that the following things occur.

(α) Any totally quantified variable of the whole pattern can be contained neither in any quantified pattern with the same variable totally quantified nor in any noun phrases with the same variable totally quantified.

(β) Any totally bound variable of the whole pattern is not totally free; any totally free variable of the whole pattern is not totally bound.

Example Ψ The pattern in example X is not a pattern-form, while that in example H is.

Note that this introduction of a pattern-form is to avoid the following types: "for any *x*, there exists *x* such that P.", and "*x* is a set, and for any *x*, P.", where P presents a pattern.

We sometimes use the notation "**P(μ)**" for a pattern-form with some totally free variable(s) **μ**. Then while we write "**P(ν)**", this presents the pattern **P(μ)** with all **μ** replaced by **ν**. When **τ** presents a noun phrase of form α, γ, or δ, "**P(τ)**" presents the pattern **P(μ)** with all **μ** replaced by **τ**. It is similar for the notations "Q(*a*,*b*)", "R(*u*, *v*, *w*)", "N(α, β, γ, δ, ε, ζ)", etc.

(E) Statement ------ a further polish of a pattern-form.

A pattern-form is said to be a statement provided that there is no totally free variable of this pattern (of course, those who are not pattern-forms are not statements). A way to get a statement is to replace all totally free variables of a pattern by certain constants. ex: "For any *x*, *x* is in *y*." is not a statement; "For any *x*, *x* is in the Gobi Desert." is a statement.

(F) Tautology

A tautology is considered as "a true statement" in common Logic, and common thoughts. However, "truth" and "false" is an undeterminable philosophical problem, so it will not be considered here. In fact, we can toughly give rules for producing tautologies. The following gives a version, which is also the most popular one.

Rule α -- Axiom method: That is, to choose a few statements, and designate them as tautologies.

Some famous examples of well-formed axiomatic systems are *ZFC* system, *NBG* system.

Example Ω The pattern in eample H is an axiom for ZFC set theory. (The empty set axiom)

Rule β -- Definition method: The production by means of "Definition", inclusive of:

(β-α)To produce the pattern "x is y" (denoted by x=y) by (C)-(c), and the pattern

$$(\forall x)(\forall y)(x=y \Leftrightarrow (\forall t)((t\in x \Leftrightarrow t\in y)\wedge(x\in t \Leftrightarrow y\in t))).$$

and then designate it as a tautology directly.

(β-β) Suppose that Q is a given pattern-form with all totally free variables η, θ,…, Ψ, and P is a new-produced pattern-form from (C)-(c) containing the same totally free variables as Q and containing some word which is not of form (a)(b), or some phrase which is an adjective phrase of form (a-a), or (b), or a verb phrase, or a noun-phrase of form (α-β), where each word or phrase has not appeared in any preceding tautology from definition method, then we can write down the new tautology[α]

$$(\forall\eta)(\forall\theta)\ldots(\forall\Psi)(P\Leftrightarrow Q)$$

(β-γ) Given pattern-form Q(w), and a new proper-noun O that has not appeared in any preceding tautology from definition method. After establishing to be a tautology the statement $(\exists !w)\ Q(w)$, we can write down the new tautology

$$(\forall w)(O = w \Leftrightarrow Q(w))$$

(β-δ) Suppose that Q presents a given pattern-form with η, θ,…, Ψ,w all its totally variables, and O presents a new-produced noun-phrase of form (α-α), (γ), or (δ) and not containing the variable w such that O = w is a pattern-form containing those totally free variables as Q, and O has not appeared in any preceding tautology from definition method. After establishing to be a tautology the statement $(\forall\eta)(\forall\theta)\ldots(\forall\Psi)(\exists !w)(Q)$, we can write down the new tautology

$$(\forall\eta)(\forall\theta)\ldots(\forall\Psi)(\forall w)(O = w \Leftrightarrow Q)$$

(β-ε) After the proper noun, the empty set "ϕ", defined, we add the method for definition: Suppose that φ is a given pattern-form with η, θ,…, Ψ, x, y, w all its totally free variables. After establishing to be a tautology the statement

$$(\forall\eta)(\forall\theta)\ldots(\forall\Psi)((S[\eta]\wedge S[\theta]\wedge S[\Omega]\wedge\ldots\wedge S[\Delta]\wedge S[\Psi])\rightarrow$$
$$(\exists !y)((\exists w)((S[y]\wedge(\forall x)(S[x]\rightarrow(x\in w\Leftrightarrow\varphi)))\wedge\ y=w)\vee$$
$$((\sim(\exists w)(\ S[y]\wedge(\forall x)(S[x]\rightarrow(x\in w\Leftrightarrow\varphi))))\wedge\ y=\phi)),$$

we can write down the new tautology

$$(\forall\eta)(\forall\theta)\ldots(\forall\Psi)(\ \{x:\varphi\}=y \Leftrightarrow$$
$$(((S[\eta]\wedge S[\theta]\wedge S[\Omega]\wedge\ldots\wedge S[\Delta]\wedge S[\Psi])\wedge S[y]\wedge(\forall x)(S[x]\rightarrow(x\in w\Leftrightarrow\varphi)))\quad\vee$$
$$((\sim(S[\eta]\wedge S[\theta]\wedge S[\Omega]\wedge\ldots\wedge S[\Delta]\wedge S[\Psi])\ \vee\ \sim(\exists w)(S[y]\wedge(\forall x)(S[x]\rightarrow(x\in w\Leftrightarrow\varphi))))\wedge y=\phi)))$$

Rule γ: Deduction method: This is a way to get tautologies throughout the "process of deduction". That is , throughout a schema for constructing a formal mathematical proof. What will be describe is a restatement of the version in the book *Set Theory for Mathematician* written by *Jean E. Rubin*. The following gives a practical example first.

<u>Example α</u> To establish the tautology "$(\exists u)(\exists v)(u=v) \to (\exists v)(\exists u)(u=v)$",

<u>*Pf*</u>	α.	[α]	$(\exists u)(\exists v)(u=v)$	(Pr)
	β.	[β]	$(\exists v)(u=v)$	(α,Pr)
	γ.	[γ]	$u=v$	(β, Pr)
	δ.	[γ]	$(\exists u)(u=v)$	(γ, EG)
	ε	[γ]	$(\exists v)(\exists u)(u=v)$	(γ, EG)
	ζ	[β]	$(\exists v)(\exists u)(u=v)$	(β, δ, ε, EP)
	η.	[α]	$(\exists v)(\exists u)(u=v)$	(α,β,ζ, EP)
	θ.	[ ]	$(\exists u)(\exists v)(u=v) \to (\exists v)(\exists u)(u=v)$	(α,ζ, CP) □

We're used to begin a process of deduction nd end this process by the notations "<u>*Pf*</u>" and "□", respectively. The process of deduction conists of several <u>*ordered steps*</u>. Each step consists of some parts:

(i) a symbol or a symbol steam representing this step.
(j) a blank bracket or a bracket on which there are some representing symbols from preceding steps.
(k) a pattern-form or a statement.
(l) Some marks to record the deductive tools.

How to write down a new line? The method are below:

[Type A]

(T) Part (i) is a symbol other than any other symbol in part (i) in preceding steps; part (j) is a blank bracket "[ ]"; part (k) is an arbitrary tautology; part (l) is "(T)"

(Pr) Part (i) is a symbol other than any other symbol in part (i) in preceding steps; part (j) is a bracket in which the symbol in part (i) is written; part (k) is an arbitrary pattern; part (l) is "(Pr)".

(MP) There're the following different steps written:

i.	S_i	P → Q	M_i
j.	S_j	P	M_j

Then we can write down a new step:

k.	$S_i + S_j$	Q	(*i*, *j*, MP)

Where $S_i + S_j$ is a bracket in which there are those symbols in S_i or in S_j.

(MT) There're the following different steps wtitten:

i.	S_i	P → Q	M_i
j.	S_j	∼Q	M_j

Then we can write down a new step:

k.	S_i+S_j	∼P	(*i*, *j*, MT)

(Simp) There're the following steps written:

i.	S_i	P and Q	M_i

Then we can write down new steps (one can choose the one needed):

j.	S_i	P	(*i*, Simp)
k.	S_i	Q	(*i*, Simp)

For convenience, one can say "From P→ Q, P, to infer Q" for MP rule, and "From P and Q, to infer P, or Q" for Simp rule. The following will be in this style.

(DS) From P or Q, ∼P, to infer Q, or, from P or Q, ∼Q, to infer P.

(A) From P, Q, to infer P and Q.

(HS) From P→Q, Q→R, to infer P→R.

(Add) From P, to infer P or Q, or, from Q, to infer P or Q.

(CD) From

i.	S_i	P→Q	M_i
j.	S_j	R→S	M_j
k.	S_k	P or R	M_k

to infer

g.	$S_i+S_j+S_k$	Q or S	(*i*, *j*, *k*, CD)

Where $S_i+S_j+S_k$ is a bracket in which there are those symbols in S_i ,in S_j ,or in S_k.

(CP) From

i.	[*i*]	P_i	M_i
j.	S_j	P_j	M_j

to infer

n.	S_j – [*i*]	$P_i \rightarrow P_j$	(*i*, *j*, CP)

where S_j – [*i*] is a bracket in which there are those symbols in S_j but not *i*.

[Type B]

Suppose that P, Q present patter-forms, then we write "P::Q" to express that we can not only "from P, to infer Q", but also "from Q, to infer P". Type B consists of rules described in this style.

(DN) P::∼∼P

(DM) ∼(P and Q):: (∼p or ∼Q); ∼(P or Q):: (∼P and ∼Q)

(Comm) (P and Q):: (Q and P); (P or Q):: (Q or P)

(Assoc) (P and Q) and R:: P and (Q and R); (P or Q) or R:: P or (Q or R)

(Dist) P and (Q or R):: (P and Q) or (P and R); P or (Q and R):: (P or Q) and (P or R) ;

(Contra) $P \to Q$ :: $\sim Q \to \sim P$

(Impl) $P \to Q$ :: $\sim P$ or Q

(Exp) (P and Q)$\to$R :: P$\to$(Q$\to$R)

(Taut) P:: (P and P); P:: (P or P)

(Equ) $P \Leftrightarrow Q$ :: (P$\to$Q) and (Q$\to$P); $P \Leftrightarrow Q$ :: (P and Q) or ($\sim$P and $\sim$Q)

[Type C]

(SI) Suppose that P(σ) presents a pattern-form with all σ variables or a noun-phrases, and P(τ) is another pattern where all σ have been replaced by τ and τ can be either a covar or a noun-phrase. As long as no free variable in σ and τ is bound in P(σ) and in P(τ), respectively, from

i.	S_i	σ = τ	M_i
j.	S_j	P(σ)	M_j

we can infer

n.	S_i+S_j	P(τ)	(*i*, *j*, SI)

(US) Suppose that P(*v*) presents a pattern-form with the variable(s) *v*, and P(τ) is another pattern where all *v* have been replaced by τ. As long as τ presents a variable or τ presents a noun-phrase in which no free variable is bound in P(τ), from

i.	S_i	(∀ *v*) P(*v*)	M_i

we can infer

n.	S_i	P(τ)	(*i*, US)

(UG) Given any symbol presenting a step, say α, we use P_α for presentation of the pattern of step α. From

i.	S_i	P_i	M_i

we can infer

n.	S_i	(∀ *v*) P(*v*)	(*i*, UG)

As long as *v* is not free in any[**α] P_x where *x* presents any symbol in the bracket S_i. and P_x cannot contain any existence-quantifier ∃ with *v* quantified.

(EG) Suppose that P(*v*) presents a pattern-form with all *v*'s totally free variable(s), and P(τ) is P(*v*) with all replaced by τ. As long as both

(α) τ presents the variable *v*, or any[**β] *v* is free in P(τ).

and (β) When τ is a noun-phrase, no free variable in τ is bound in P(τ).

Then from

i.	S_i	P(τ)	M_i

we can infer

n.	S_i	(∃ *v*) P(*v*)	(*i*, EG)

(EP) With the same convention of P_α in (UG), As long as v is not free in any P_x where x is in the bracket S_k but x doesn't present j, from

$i.$	S_i	$(\exists v)\ P_j$	M_i
$j.$	$[j]$	P_j	M_i
$k.$	S_k	P_k	M_i

we can infer

$n.$	$(S_i+S_j)-[j]$	P_k	(i, j, k, EP)

where $(S_i+S_j)-[j]$ is a bracket in which there're those symbols in S_i or in S_j but not j.

(Uni) Suppose that $P(\mu)$ presents a given pattern-form in which each μ is free and there is no v, and $P(v)$ presents a pattern where all μ have been replaced by v, then we have

$$(\exists !\mu)\ P(\mu) \quad :: \quad (\exists \mu)(P(\mu) \text{ and } (\forall v)(P(v) \rightarrow \mu = v))$$

Finally, while there is some step with its bracket blank, and with its pattern a statement, then this statement becomes a tautology.

[Note] (**α) When S_i contains nothing, then we contract that v is not free in any $^{**\alpha}$ P_x.
(**β) When $P(\tau)$ contains no v, then we contract that any $^{**\beta}$ v is free in $P(\tau)$.
It is similar for other cases.

(A rule for a presented pattern)

Additionally, for every presented pattern, say *P*, we can perform the deduction process directly on *P* as long as this presented pattern satisfies the conditions for those required deduction tools. For example, suppose that P presents a pattern-form, then the establishment of the tautology "$(\exists u)(\exists v)P \rightarrow (\exists v)(\exists u)P$" is similar as that above, but we have to substitute P for any "$u = v$". After this process, we can do the following:

From such a given pattern-form, to get the tautology "$(\exists u)(\exists v)P \rightarrow (\exists v)(\exists u)P$".

(G) Other notations.

Every new-introduced tautology has to be labelled according to how it is produced. When a tautology is labelled "**<u>Axiom</u>**", it is produced from the axiom method; when it is labelled "**<u>Definition</u>**", it's produced by definition method; when it is labelled "**<u>Theorem</u>"**, it's produced by deduction method.

For example, further in set theory, we will see that:

" **<u>Theorem 4</u>**

There uniquely exists *B* such that "*B* is a set and for any *x*, if *x* is a set, then not *x* is in *B*. "

Here, it is labelled "Theorem". We can see that it's given according to the deduction method.

<u>Example β</u> Suppose that *u*, *v* are the whole variables in pattern forms presented by *P*. Establish that

(a) $(\exists u)(\exists v)\, P \rightarrow (\exists v)(\exists u)\, P$

(b) $(\forall u)(\forall v)\, P \rightarrow (\forall v)(\forall u)\, P$

(c) $(\exists u)(\forall v)\, P \rightarrow (\forall v)(\exists u)\, P$

Since *(a)* is mentioned, we only give the proof of *(b)* and *(c)*.

Proof of (b)

*.	[*]	$(\forall u)(\forall v)\, P$	(Pr)
***.	[*]	$(\forall v)\, P$	(*, US)
+.	[*]	P	(***, US)
^^.	[*]	$(\forall u)\, P$	(*+*, UG)
**.	[*]	$(\forall v)(\forall u)\, P$	(^^, UG)
^__^.	[ ]	$(\forall u)(\forall v)\, P \rightarrow (\forall v)(\forall u)\, P$	(*,**, CP) □

Proof of (c)

C.	[C]	$(\exists u)(\forall v)\, P$	(Pr)
CC.	[CC]	$(\forall v)\, P$	(Pr)
CCC.	[CC]	P	(CC, US)
CD.	[CC]	$(\exists u)\, P$	(CCC, EG)
D.	[CC]	$(\forall v)(\exists u)\, P$	(CD, UG)
DC.	[C]	$(\forall v)(\exists u)\, P$	(C,CC,D, EP)
DCC.	[ ]	$(\exists u)(\forall v)\, P \rightarrow (\forall v)(\exists u)\, P$	(C,DC, CP) □

Example γ Suppose that u, v are the whole variables in pattern forms presented by P. Establish that

(a) $(\forall u)(\forall v)\, (\sim P \vee P)$

(b) $(\forall u)(\forall v)\, (\sim(P \wedge \sim P))$

Proof of (a)

S.	[S]	P	(Pr)
T.	[S]	$P \wedge P$	(S, taut)
U.	[S]	P	(T, Simp)
V.	[ ]	$P \rightarrow P$	(S,U, CP)
W.	[ ]	$\sim P \vee P$	(V, Impl)
X.	[ ]	$(\forall v)\, (\sim P \vee P)$	(W, UG)
Y.	[ ]	$(\forall u)\, (\forall v)\, (\sim P \vee P)$	(X, UG) □

Proof of (b)

S.	[S]	P	(Pr)
T.	[S]	$\sim\sim P$	(S, DN)
U.	[S]	$P \rightarrow \sim\sim P$	(S,T, CP)
V.	[ ]	$\sim P \vee \sim\sim P$	(U, Impl)
W.	[ ]	$\sim(P \wedge \sim P)$	(V, DM)
X.	[ ]	$(\forall v)\, (\sim(P \wedge \sim P))$	(W, UG)

Y. [ ] $(\forall u)(\forall v)(\sim(\mathrm{P}\wedge\sim\mathrm{P}))$ (X, UG) □

Example δ Suppose that P(u) is a pattern with u totally free and u, r do not appear in P(u). Establish the following

a) $(\exists u)\,\mathrm{P}(u) \to (\exists v)\,\mathrm{P}(v)$.

b) $(\forall u)\,\mathrm{P}(u) \to (\forall v)\,\mathrm{P}(v)$.

c) $(\exists! u)\,\mathrm{P}(u) \to (\exists! v)\,\mathrm{P}(v)$.

Proof of a

	i.	[*i*]	$(\exists u)\,\mathrm{P}(u)$	(Pr)	
	j.	[*j*]	$\mathrm{P}(u)$	(Pr)	
	k.	[*j*]	$(\exists v)\,\mathrm{P}(v)$	(*j*, EG)	
	m.	[*i*]	$(\exists v)\,\mathrm{P}(v)$	(*i*,*j*,*k*, EP)	
	n.	[ ]	$(\exists u)\,\mathrm{P}(u) \to (\exists v)\,\mathrm{P}(v)$.	(*i*,*m*, CP)	□
Proof of b	*s*.	[*s*]	$(\forall u)\,\mathrm{P}(u)$	(Pr)	
	t.	[*s*]	$\mathrm{P}(v)$	(*s*, US)	
	u.	[*s*]	$(\forall v)\,\mathrm{P}(v)$	(*t*, UG)	
	v.	[ ]	$(\forall u)\,\mathrm{P}(u) \to (\forall v)\,\mathrm{P}(v)$	(*i*,*m*, CP)	□
Proof of c	A.	[A]	$(\exists! u)\,\mathrm{P}(u)$	(Pr)	
	B.	[B]	$(\exists u)\,(\mathrm{P}(u)\wedge(\forall r)(\mathrm{P}(r)\to r=u))$	(A, Uni)	
	Γ.	[Γ]	$\mathrm{P}(u)\wedge(\forall r)(\mathrm{P}(r)\to r=u)$	(Pr)	
	Δ.	[Γ]	$(\exists v)\,(\mathrm{P}(v)\wedge(\forall r)(\mathrm{P}(r)\to r=v))$	(Γ, EG)	
	E.	[Γ]	$(\exists! v)\,\mathrm{P}(v)$	(Δ, Uni)	
	Z.	[A]	$(\exists! v)\,\mathrm{P}(v)$	(B,Γ,E, EP)	
	H.	[ ]	$(\exists! u)\,\mathrm{P}(u) \to (\exists! v)\,\mathrm{P}(v)$	(A,Z, CP)	□

Note that when a bracket contains more than a single mark, then in this bracket, they have to be separated by commas ",". Please see the example:

Example ε Establish that "$(\forall a)(\,((\exists b)(a{=}b)\wedge(\exists c)(a\in c)) \to (\exists b)(\exists c)(b\in c)\,)$"

Proof

I.	[I]	$(\exists b)(a{=}b)\wedge(\exists c)(a\in c)$	(Pr)
II.	[I]	$(\exists b)(a{=}b)$	(I, Simp)
III.	[I]	$(\exists c)(a\in c)$	(I, Simp)
IV.	[IV]	$a=b$	(Pr)
V.	[V]	$a\in c$	(Pr)
VI.	[IV,V]	$b\in c$	(IV,V, SI)
VII.	[IV,V]	$(\exists c)(b\in c)$	(VI, EG)
VIII.	[IV,V]	$(\exists b)(\exists c)(b\in c)$	(VII, EG)
IX.	[IV,V]	$(\exists b)(\exists c)(b\in c)$	(III,V,VIII, EP)
X.	[I]	$(\exists b)(\exists c)\,(b\in c)$	(II,IV,IX EP)

XI.	[ ]	$((\exists b)(a{=}b)\wedge(\exists c)(a\in c)) \rightarrow (\exists b)(\exists c)(b\in c)$	(I,X, CP)
XII	[ ]	$(\forall a)(((\exists b)(a{=}b)\wedge(\exists c)(a\in c)) \rightarrow(\exists b)(\exists c)(b\in c))$	(XI, UG) □

Now, we're going to establish some theorems directly from logic.

Theorem *a* $(\forall x)(x=x)$

Theorem *b* $(\forall x)(\forall y)(x=y \Leftrightarrow y=x)$

Theorem *c* $(\forall x)(\forall y)(\forall z)((x=y$ and $y=z) \rightarrow x=z)$

We only give the *proof* of *a.*, while the others are left.

Proof

I.	[I]	$t\in x$	(Pr)
II.	[I]	$t\in x$ and $t\in x$	(I, taut)
III.	[I]	$t\in x$	(II, Simp)
IV.	[ ]	$t\in x \rightarrow t\in x$	(I,III, CP)
V.	[ ]	$(t\in x \rightarrow t\in x)$ and $(t\in x \rightarrow t\in x)$	(IV, taut)
VI.	[ ]	$t\in x \Leftrightarrow t\in x$	(V, equ)
VII.	[VII]	$x\in t$	(Pr)
VIII.	[VII]	$x\in t$ and $x\in t$	(VII, taut)
IX.	[VII]	$x\in t$	(VII, IX, CP)
X.	[ ]	$x\in t \rightarrow x\in t$	(VII, IX, CP)
XI.	[ ]	$(x\in t \rightarrow x\in t)$ and $(x\in t \rightarrow x\in t)$	(X, taut)
XII.	[ ]	$x\in t \Leftrightarrow x\in t$	(XI, equ)
XIII.	[ ]	$(t\in x \Leftrightarrow t\in x)$ and $(x\in t \Leftrightarrow x\in t)$	(VI, XII, Conj)
XIV.	[ ]	$(\forall t)((t\in x \Leftrightarrow t\in x)$ and $(x\in t \Leftrightarrow x\in t))$	(XIII, UG)
XV.	[ ]	$(\forall x)(\forall y)(x{=}y \Leftrightarrow (\forall t)((t\in x \Leftrightarrow t\in y)\wedge(x\in t\Leftrightarrow y\in t)))$	(T)
XVI.	[ ]	$(\forall y)(x{=}y \Leftrightarrow (\forall t)((t\in x \Leftrightarrow t\in y)\wedge(x\in t\Leftrightarrow y\in t)))$	(XV, US)
XVII.	[ ]	$x{=}x \Leftrightarrow (\forall t)((t\in x \Leftrightarrow t\in x)\wedge(x\in t\Leftrightarrow x\in t))$	(XVI, US)
XVIII.	[ ]	$(x{=}x \rightarrow (\forall t)((t\in x \Leftrightarrow t\in x)\wedge(x\in t\Leftrightarrow x\in t)))$ and	
XIX.	[ ]	$(\forall t)((t\in x \Leftrightarrow t\in x)\wedge(x\in t\Leftrightarrow x\in t)) \rightarrow x=x$	(XVIII, Simp)
XX.	[ ]	$x=x$	(XIV, XIX, MP)
XXI.	[ ]	$(\forall x)(x=x)$	(XX, UG) □

Example ζ Establish that $(\forall a)(\exists b)(b=a)$.

pf

I.	[ ]	$(\forall a)(a=a)$	(T)
II.	[ ]	$a=a$	(I, US)
III.	[ ]	$(\exists b)(b=a)$	(II, EG)
IV.	[ ]	$(\forall a)(\exists b)(b=a)$	(III, UG)

[Note α] If somebody thinks that those "…" make the rule (γ-γ) not clear, well, one can extend this rule in detail. That is, one may write down several versions, as follow:

$(\forall \eta)(P \Leftrightarrow Q)$	for the case with only η.
$(\forall \eta)(\forall \theta)(P \Leftrightarrow Q)$	for the case with η,θ.
$(\forall \eta)(\forall \theta)(\forall \iota)(P \Leftrightarrow Q)$	for the case with η,θ,ι.
$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(P \Leftrightarrow Q)$	for the case with η,θ,ι,κ.
$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(P \Leftrightarrow Q)$	for the case with η,θ,ι,κ,λ.
$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(P \Leftrightarrow Q)$	for the case with η,θ,ι,κ,λ,μ.
$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(P \Leftrightarrow Q)$	for the case with η,θ,ι,κ,λ,μ,ν.
$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(P \Leftrightarrow Q)$	for the case with η,θ,ι,κ,λ,μ,ν,ξ.

and additionally

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(\forall \text{Ж})(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(\forall \text{Ж})(\forall \sigma)(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(\forall \text{Ж})(\forall \sigma)(\forall \tau)(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(\forall \text{Ж})(\forall \sigma)(\forall \tau)(\forall \text{Ш})(P \Leftrightarrow Q)$

$(\forall \eta)(\forall \theta)(\forall \iota)(\forall \kappa)(\forall \lambda)(\forall \mu)(\forall \nu)(\forall \xi)(\forall o)(\forall \pi)(\forall \rho)(\forall \text{Ж})(\forall \sigma)(\forall \tau)(\forall \text{Ш})(\forall \text{Ю})(P \Leftrightarrow Q)$

for the case with η,θ,ι,κ,λ,μ,ν,ξ,ο, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ,Ж, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ,Ж,σ, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ,Ж,σ,τ, with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ,Ж,σ,τ,Ш and with η,θ,ι,κ,λ,μ,ν,ξ,ο,π,ρ,Ж,σ,τ,Ш, and Ю, repectively.

But now, another question arises. How to deal with the case while the variables are over the given ones. A way is to abandon this system and rebuild another logical system, with a version with that many variables.

[Note γ] Perhaps somebody may think that the restriction of phrases, pattern forms, and rules for tautologies especially for formal proof are over detail, formalized, unintuitive, and stuffy. However, it gives a concretization of such a criterion from which we perform logical thinking, and is a description of the model of the whole logical system that operates basically and independently to support such a tremendous mathematical system silently. Maybe having a look on the background work is of no damage for a pure mathematician.

Despite the fact that a formal proof of a theorem contains complete information, mathematicians and logicians still prefer to gibe what is called an informal proof. It is a form consisting of more liberty in organization about sentences about expression and more intuition in the problem itself. The following is a reproof of example ε by informal proof.

<u>Example η</u> Reprove example ε.

Reproof

Let a be given. By condition we can find a b such that $a = b$, and a c such that $a \in c$, hence we've found the b and c such that $b \in c$, and then this statement

$$(\forall a)(((\exists b)(a=b) \wedge (\exists c)(a \in c)) \rightarrow (\exists b)(\exists c)(b \in c))$$

is proved. □

Sometimes, we even abandon the formal types of definition, theorem, and axiom, and choose a more liberal type, the informal type. For instance,

"Let X, Y be sets. X, Y are said to be disjoint if $X \cap Y = \phi$"

is a rewrite of that

"X and Y are disjoint $\Leftrightarrow$ $((\mathrm{S}[X] \wedge \mathrm{S}[Y]) \wedge X \cap Y = \phi)$"

with the quantifiers $(\forall x)$, $(\forall y)$ omitted, (Note that the linking-verb "are" has to be replaced by the word "is" since we have to view the word "and" as a preposition. However, due to the custom in English writing, using the word "are" leads to no confusion.) and that

Example ι

"Let A, B be sets. The Cartesion Product $A \times B$ is defined as the set $\{<u, v>: u \in A$ and $v \in B\}$."

is a rewrite of that

"$A \times B = P \Leftrightarrow (((\mathrm{S}[A] \wedge \mathrm{S}[B] \wedge P=\{<u, v>: u \in A \text{ and } v \in B\}) \vee (\sim(\mathrm{S}[A] \wedge \mathrm{S}[B]) \wedge P=\phi))$"

With the quantifiers $(\forall A)$, $(\forall B)$ and $(\forall P)$ omitted.

Example κ

"Let A, B, C be sets. Then $P(A, B, C)$.", "For any sets A, B, C, $P(A, B, C)$", and "For any A, B, C being sets, $P(A, B, C)$." are rewrite of that

"$(\forall A)(\forall B)(\forall C)(((\mathrm{S}[A] \wedge \mathrm{S}[B]) \wedge \mathrm{S}[C]) \rightarrow P(A, B, C))$".

"There exists sets a, b, and relations c, d such that $Q(a, b, c, d)$" is a rewrite of that

"$(\exists a)(\exists b) (\exists c) (\exists d)(((\mathrm{S}[a] \wedge \mathrm{S}[b]) \wedge (\mathrm{Rel}[c] \wedge \mathrm{Rel}[d])) \wedge Q(a, b, c, d))$".

Example λ

"For any $\varepsilon>0$, there is a $\delta>0$ such that for any x with $|x-1|<\delta$, we always have $|f(x)-3|<\varepsilon$", and that described by symbols "$\forall \varepsilon>0, \ \exists \delta>0$, s.t. $\ \forall |x-1|<\delta, |f(x)-3|<\varepsilon$" is a rewrite of

$$(\forall \varepsilon)(\varepsilon>0 \rightarrow (\exists \delta)(\delta>0 \wedge (\forall x)(|x-1|<\delta \rightarrow |f(x)-3|<\varepsilon))),$$

Where, this statement is a topic further in Calculus. In mathematical analysis,

"A is d-open in M."

is a rewrite of that "A is open in M with respect to d.", since we don't introduce the form "variable-adjective" as formal type but we have the sense of the meaning.

Example μ

The definition in Number Theory "Let a, b be integers, a divides b if there's an integer c such that $b = ac$." Is a rewrite of "a divides b if and only if a is an integer and b is an integer and there exists c such that c is an integer and $b = ac$"."

In further developments of other area in mathematics, formal description will be changed to informal description.

[Note]

Combined methods

In proofs of some previous examples we can easily notice that to achieve some goals, we have certain tricks. Hence we may combine these tricks together and obtain new method for deduction process in order to simplify our proof. For example,

Example ν To obtain SF:

(SF) From	J.	S_J	P	M_J
To infer	L.	S_J	P	(J, SF)
[STUDY]	J.	S_J	P	M_J
	K.	S_J	$P \wedge P$	(J, taut)
	L.	S_J	P	(K, Simp)

Example ξ To obtain SB:

(SB) Suppose that P(u) is a pattern-form with all u's totally free and v does not appear in P(u). Then from $(\exists u)$ P(u) we can infer $(\exists u)$ P(u), from $(\forall u)$ P(u) we can infer $(\forall v)$ P(v), and from $(\exists! u)$ P(u) we can infer $(\exists! v)$ P(v).

This method can be similarly studied from the proof of example δ.

Example ο To obtain AIF:

(AIF)	From	J.	S_J	$Q \rightarrow R$	M_J
	To infer	T.	S_J	$(P \wedge Q) \rightarrow R$	(J, AIF)
	[**STUDY**]				

J.	S_J	$Q \rightarrow R$	M_J
K.	S_J	$\sim Q \vee R$	(J, Impl)
L.	S_J	$\sim P \vee (\sim Q \vee R)$	(K, Add)
M.	S_J	$(\sim P \vee \sim Q) \vee R$	(L, Asso)
X.	[X]	$\sim P \vee \sim Q$	(Pr)
W.	[X]	$\sim (P \wedge Q)$	(X, Dist)
V.	[ ]	$(\sim P \vee \sim Q) \rightarrow \sim (P \wedge Q)$	(X,W, CP)
U.	S_J	$\sim (P \wedge Q) \vee R$	(M,V, CD)
T.	S_J	$(P \wedge Q) \rightarrow R$	(U, Impl)

This illustration is **STUDY** is merely an explanation of this formalized method. It might not be a proof. After all, we cannot caearly give a way about what to show. We can only convince ourselves this combination does make sense. We arenot forced to accept. Those who doesn't not admit these methods might ignore them. The following gives more extra tools, with the **STUDY**'s omitted.

Example π To obtain DUS, TUS, and QUS:

(DUS)	From $(\forall u)(\forall v)\ P(u,v)$	to infer $P(\upsilon, \varphi)$
(TUS)	From $(\forall u)(\forall v)(\forall w)\ P(u,v,w)$	to infer $P(\upsilon, \varphi, \chi)$
(QUS)	From $(\forall a)(\forall b)(\forall c)(\forall d)\ P(a,b,c,d)$	to infer $P(\alpha, \beta, \gamma, \delta)$

Example ρ To obtain SIF:

(SIF)	From P, $(P \wedge Q) \rightarrow R$,	to infer $Q \rightarrow R$

Example σ To obtain SPr:

(SPr)	From $(\exists v)\ P(v)$	to infer $P(v)$

The following is an example containing steps with AIF, DUS, SIF, or SPr.

Example τ Suppose that $P(u)$ presents a pattern-form containing neither v, w nor r, establish that

a) $(\exists! u)\ P(u) \rightarrow ((\exists u)\ P(u) \wedge (\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w))$

b) $((\exists u)\ P(u) \wedge (\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w)) \rightarrow (\exists! u)\ P(u)$

proof of a

I.	[I]	$(\exists! u)\ P(u)$	(Pr)
II.	[I]	$(\exists u)\ (P(u) \wedge (\forall v)(P(v) \rightarrow u = v))$	(I, Uni)
III.	[III]	$P(u) \wedge (\forall v)(P(v) \rightarrow u = v)$	(Pr)
IV.	[III]	$P(u)$	(III, Simp)
V.	[III]	$(\exists u)\ P(u)$	(IV, EG)
VI.	[I]	$(\exists u)\ P(u)$	(II,III,V, EP)
VII.	[III]	$(\forall v)(P(v) \rightarrow u = v)$	(III, Simp)
VIII.	[III]	$P(w) \rightarrow v = w$	(VII, US)
IX.	[III]	$(P(v) \wedge P(w)) \rightarrow v = w$	(VIII, AIF)
X.	[I]	$(P(v) \wedge P(w)) \rightarrow v = w$	(II,III,IX, EP)
XI.	[I]	$(\forall w)((P(v) \wedge P(w)) \rightarrow v = w)$	(X, UG)
XII.	[I]	$(\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w)$	(XI, UG)
XIII.	[I]	$(\exists u)\ P(u) \wedge (\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w)$	(VI,XII, Add)
XIV.	[ ]	$(\exists! u)\ P(u) \rightarrow ((\exists u)\ P(u) \wedge (\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w))$	(I, XIII, CP) □

proof of b

I.	[I]	$((\exists u)\ P(u) \wedge (\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w))$	(Pr)
II.	[I]	$(\exists u)\ P(u)$	(I, Simp)
III.	[I]	$P(u)$	(II, SPr)
IV.	[I]	$(\forall v)(\forall w)((P(v) \wedge P(w)) \rightarrow v = w)$	(I, Simp)
V.	[I]	$(P(u) \wedge P(v)) \rightarrow u = v$	(IV, DUS)
VI.	[I]	$P(v) \rightarrow u = v$	(V, SIF)
VII.	[I]	$(\forall v)(P(v) \rightarrow u = v)$	(VI, UG)
VIII.	[I]	$P(u) \wedge (\forall v)(P(v) \rightarrow u = v)$	(III,VII, A)

IX.	[I]	$(\exists u)\,(\mathrm{P}(u) \wedge (\forall v)(\mathrm{P}(v)\rightarrow u = v))$	(VIII, EG)
X.	[I]	$(\exists! u)\,\mathrm{P}(u)$	(IX, Uni)
XI.	[ ]	$((\exists u)\,\mathrm{P}(u) \wedge (\forall v)(\forall w)((\mathrm{P}(v)\wedge\mathrm{P}(w))\rightarrow v = w)) \rightarrow (\exists! u)$	(I,X, CP) □

We might intuitively say that the pattern "$(\forall v)(\forall w)((\mathrm{P}(v)\wedge\mathrm{P}(w))\rightarrow v = w)$" presents the uniqueness of u. That is, "this u such that $\mathrm{P}(u)$ is unique". Then we can find that to show $(\exists! u)\,\mathrm{P}(u)$, it suffices to show the existence and uniqueness of u.

Frequently in set theory, and even in mathematics, a preparing uniqueness theorem for a definition of the type (β - δ) is in a modified form

$$(\forall v_\alpha)(\forall v_\alpha)\ldots(\forall v_s)(\mathrm{P}(v_\alpha, v_\alpha, \ldots, v_s)\rightarrow(\exists! w)\,\mathrm{Q}(v_\alpha, v_\alpha, \ldots, v_s, w)),$$

which would make sense because we can prove the following property

$$(\forall v_\alpha)(\forall v_\alpha)\ldots(\forall v_s)(\mathrm{P}(v_\alpha, v_\alpha, \ldots, v_s)\rightarrow(\exists! w)\,\mathrm{Q}(v_\alpha, v_\alpha, \ldots, v_s, w)) \Leftrightarrow$$
$$(\forall v_\alpha)(\forall v_\alpha)\ldots(\forall v_s)\,(\exists! w)$$
$$((\mathrm{P}(v_\alpha, v_\alpha, \ldots, v_s)\wedge\mathrm{Q}(v_\alpha, v_\alpha, \ldots, v_s, w))\vee(\sim\mathrm{P}(v_\alpha, v_\alpha, \ldots, v_s)\wedge w=\phi)$$

Similarly, the notation "ϕ" presents the empty set in the assumption that it's been defined.

In fact, there are some common informal forms of the writing of a definition in type (β-δ). After establishing the theorem "Let x, y, z, be sets such that $\mathrm{P}(x, y, z)$. If $\mathrm{Q}(x, y, z)$ then $\exists! w$ such that $\mathrm{R}(w)$ and $\mathrm{S}(x, y, z, w)$", we can write down a definition. The following forms mean the same.

(ғ) $\mathrm{O}(x, y, z) = w \Leftrightarrow (\mathrm{P}(x,y,z)\wedge\mathrm{Q}(x,y,z)\wedge\mathrm{R}(w)\wedge\mathrm{S}(x,y,z,w))\vee(\sim(\mathrm{P}(x,y,z)\wedge\mathrm{Q}(x,y,z))\wedge w=\phi)$

(ж) Let x, y, z be sets such that $\mathrm{P}(x,y,z)$. If $\mathrm{Q}(x,y,z)$, then define

$\mathrm{O}(x, y, z) = w \Leftrightarrow (\mathrm{Q}(x,y,z)\wedge\mathrm{R}(w)\wedge\mathrm{S}(x,y,z,w))\vee(\sim\mathrm{Q}(x,y,z)\wedge w=\phi)$

(к) Let Let x, y, z be sets such that $\mathrm{P}(x,y,z)$ and $\mathrm{Q}(x,y,z)$. Then define

$\mathrm{O}(x, y, z) = w \Leftrightarrow (\mathrm{R}(w)\wedge\mathrm{S}(x,y,z,w))$

(н) Let x, y, z be sets such that $\mathrm{P}(x,y,z)$, $\mathrm{Q}(x,y,z)$, and $\mathrm{R}(w)$. Then define $\mathrm{O}(x, y, z) = w \Leftrightarrow \mathrm{S}(x,y,z,w)$

[Note] In fact, "S[x]" and "x is in y" are basic pattern for ZFC axiomatic system. In general, in our logica system these basic patterns can be replaced. One may create by construction method (c-c) several new patterns (Suppose that they re P_{I}, P_{II}, $\mathrm{P}_{\mathrm{III}}$, P_{IV}, P_{V}, are created.). If required, one may create some phrases, some proper nouns as basic forms. Note that these basic forms course the requirement of changing some definition methods of producing tautologies (method (β-α), and (β-δ) has to be changed) since they are not compounded by changed basic pattern P_{I} to P_{V})

Making such generalization is due to the fact that we may construct another axiomatic system. For example, whenever we rewrite the thoughts of Euclid about Classic Geometry, we may write down quite a few basic patterns, inclusive of "P is a point", "L is a line.", "M and N are perpendicular.", etc.

Peano's Axiomatic system is an axiomatic system for natural numbers. It contains a basic prop noun "0" and basic patterns "x is a natural number", x is equal to y", "y is a successor of x".

NBG Axiomatic system is another one consisting of basic patterns "X is an atom", "X is a class”, and “*X* is in *Y*”. The concept “class” is a polish of “sets” in ZFC set theory.

The Field axiom, Ordering axiom, and Completeness axiom together is a large axiomatic system for the beginning of real number. It has a few basic patterns, “x is real”, “The sum of x and y is z”, “The product of x and y is z”, “x is larger than y”, etc. However, the statement of Completeness axiom still needs the concept of sets. It might not be a independent system, or I have to say that it is attached to ZFC system.

There is an interesting thing. Begin with ZFC system, and proceed by logical method for a while, one ma define “0”, a “natural number”, the concept “successor” by definition method, and then prove the Peano axioms and collect them as theorems. Moreover, under sufficient development, one may define those basic patterns for real number system and prove these combined axioms as theorems. In fact, some textbooks of Number Theory and some of Calculus would start their content with Peano axiom and Field, Ordering, Completeness axiom, respectively, in order to prevent detail not concerning their chapters.

[Note] In language system for mathematic, we only focus on producing new statements (tautologies) through logical rules, and do not care about truth and falsity of statements. We might even accept the possibility of getting a tautology “P and not P” from a pattern P. In fact, the standatd of truth and falsity and the absurdness of “P and not P” come from our feeling, experience, and NOBODY said that we MUST prevent those cases absolutely, isn’t it?
(Concerning further discussion of statements, tautology, truth, falsity, and prevention of “P and not P”, etc, they are all considered in theories called symbolic logic and mathematical logic, branches of mathematic, and both studied based on Set Theory.)

[Note] It might be easy to notice that in all discuss above, we are constructing a language system by another language system. Indeed, we are doing so and we now have to distinguish these systems. The language used to describe mathematical properties (what we introduce) is called an object-language, while the language used to give rules for this object-language (what we’re using) is called a meta-language. We might imagine that we could have a meta-meta-language, a meta-meta-meta-language,…,etc. Similarly, we can introduce a language due to object-language, and this new language is then called a sub-language. We could easily imagine consequently we could have a sub-sub-language, a sub-sub-sub-language, etc.

The comprehension of the mathematical-language is based on the comprehension of rules of this language.While the rules are sritten in English, they are based on English. Moreover, the ecomprehension of English depends on experience and familiarity of this language. We have to believe that people (English-speakers) are able to deliver messages through English absolutely precisely. Otherwise, without such a common language, English, how do they talk? How to thdy read an article? And how do they discuss mathematics?

Have we naturally accepted this fundamental cognition? Yes!

The collection of conventions is a summarization, which can be viewed as a list about how we think. It can be applied to any axiomatic system, such as mathematical system, or a thinking system of a person. In ZFC system, it will be applied to its axioms for set theory.

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