# Cardinality

W.S.P

May 2, 2012

To consider the "size" of a set, the most important notion is the "functions". If $f : A \to B$ is a function, there are some thoughts that we ought to know:

**1** (One-to-one)**.** *Supposely that whenever $f(x) = f(y)$, it follows that $x = y$, then the function $f$ is called one-to-one from $A$ to $B$.*

**2** (Onto)**.** *Supposely that for each $b \in B$, there is an $a \in A$ such that $f(a) = b$, then this function $f$ is called from $A$ onto $B$.*

**3.** *An Injection (injective function) is an one-to-one function; a surjection (surjective function) is an onto function; a bijection (bijective function) is an injective and surjective function.*

**4** (Equinumerosity)**.** *The sets $A$, $B$ are called equinumerous (of the same cardinality as) (Denoted by $A \approx B$) if there is a bijection from $A$ to $B$; $A \preceq B$ (of weakly less cardinality than) if there is an injection from $A$ to $B$. $A \prec B$ (of (strickly) less cardinality than) if $A \preceq B$ but $A \not\approx B$.*

Now I think it is necessary to verify that if those elementary functions are one-to-one and onto between each pair of given sets.

**5.** *$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \pi x$ is one-to-one and onto because,*

*(i) For $x, y \in \mathbb{R}$, if $f(x) = f(y)$, i.e. $\pi x = \pi y$, then $x = y$ immediately.*

*(ii) For $y_0 \in \mathbb{R}$, choose $x = \frac{y_0}{\pi}$. Then $f(x) = \pi \cdot \frac{y_0}{\pi} = y_0$.*

**6.** *$g : \mathbb{R}^+ \to \mathbb{R}^+,\ x \to x^2$ is a bijective.*

*(i) For $a, b \in \mathbb{R}^+$, if $a^2 = b^2$ then $(a-b)(a+b) = 0$. Because $a + b \neq 0$, we obtain $a - b = 0$, i.e. $a = b$.*

*(ii) For $b_0 \in \mathbb{R}^+$, we choose $a = \sqrt{b_0}$. Then $g(a) = (\sqrt{b_0})^2 = b_0$.*

*Hence, $g$ is bijective.*

**7.** *The function $u(x) = x^3$ in $\mathbb{R}$ is one-to-one and onto.*

*Proof.* For $y \in \mathbb{R}$, choose $x = \sqrt[3]{y}$. This implies that $u$ is onto. If $x, y \in \mathbb{R}$ such that $x^3 = y^3$. Then consider

$$\begin{aligned} 0 = x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\ &= (x-y)\left((x + \frac{1}{2}y)^2 + \frac{3}{4}y^2\right) \end{aligned}$$

If $(x-y)\left((x+\frac{1}{2}y)^2 + \frac{3}{4}y^2\right) = 0$, then $x = y$. $(=0)$. If $(x-y)\left((x+\frac{1}{2}y)^2 + \frac{3}{4}y^2\right) \neq 0$, then $x - y = 0$. Hence $u$ is one-to-one. □

Another way to show "one-to-one" is that we have several cases according as $x>0$, $x = 0$, or $x<0$ and, $y>0$, $y = 0$, or $y<0$. For example, if $x, y$ both $>0$ ($<0$) then $x^2 + xy + y^2 > 0$. Hence $x^3 - y^3 = 0$ implies $x = y$.

This division inspires us a helpful property.

**8.** *If $A \cap B = C \cap D = \varnothing$, and $f : A \cup B \to C \cup D$ is*

*(i) one-to-one from $A$ onto $C$ and*

*(ii) one-to-one from $B$ onto $D$,*

*then $f$ is a bijection.*

*This is a trivial statement, so we omit the proof.*

Still another example is a trigonometric function.

**9.** *Show that $G(x) = \sin x$, where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is one-to-one from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(-1, 1)$.*

*Proof.*

(one-to-one) If $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $G(x) = G(y)$. Then

$$0 = \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

Since $\frac{x+y}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos\left(\frac{x+y}{2}\right) \neq 0$, it follows that

$$\sin\left(\frac{x-y}{2}\right) = 0.$$

Hence $x = y$.

(onto) Let $K \in (-1, 1)$. Since

(i) $G(x) = \sin x$ is continuous in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\sin(-\frac{\pi}{2}) = -1$, $\sin(\frac{\pi}{2}) = 1$.

(iii) $-1<K<1$.

Intermediate Value Theorem hence tell us that there is a $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$G(c) = \sin c = K.$$

$\square$

Now our purpose is to show that $\mathbb{N} \approx \mathbb{Z} \approx \mathbb{Q} \prec \mathbb{R} \approx \mathbb{C}$.

**10.** $\mathbb{N} \approx \mathbb{Z}$

*Proof.* We ought to find a bijection. Let

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ -\frac{n-1}{2}, & \text{otherwise.} \end{cases} \tag{1}$$

Then it is routine to show bijectivity.

For even natural numbers $x, y$, if $f(x) = f(y)$, i.e. $\frac{x}{2} = \frac{y}{2}$, then $x = y$. Next, for $p \in \mathbb{Z}$, $p>0$, we find that $f(2p) = \frac{2p}{2} = p$; For odd natural numbers $x, y$, if $f(x) = f(y)$, i.e. $-\frac{x-1}{2} = -\frac{y-1}{2}$, then $x = y$. Similarly, if $p \in \mathbb{Z}$, $p \leq 0$, then $f(-2p+1) = -\frac{(-2p+1)-1}{2} = p$. By previous example, $f$ is bijective. $\square$

For the goal of the fact that $\mathbb{Z} \approx \mathbb{Q}$, we need quite a few effort. Firstly, we embed a significant property into our discussion.

**11.** *Let $\Lambda$ be an index set. Given $f : \sqcup_{j\in\Lambda} A_j \to \sqcup_{j\in\Lambda} C_j$, if for any $j \in \Lambda$, $f$ is one-to-one from $A_j$ onto $C_j$, then $f$ is one-to-one and onto.*

*Note that the notation $\sqcup_{j\in\Lambda} S_j$ means disjoint union. If the sets $S_j$'s are pairwise disjoint, we write $\sqcup_{j\in\Lambda} S_j$ for their union instead.*

The statement is useful in the following property, by which we will show that $\mathbb{N} \approx \mathbb{Q}^+$.

**12.** $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$.

*Proof.* Define

$$\widehat{n} = \begin{cases} 0, & \text{if } n \leq 0,\ n \in \mathbb{Z}; \\ 1 + 2 + 3 + \cdots + n, & \text{if } n \in \mathbb{N}. \end{cases} \tag{2}$$

Let $p(m, n) = \widehat{m + n - 2} + m$. We're going to show that $p$ gives a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. If $m + n = k \in \mathbb{N}$, we hope to verify that $f$ is bijective from

$$A_k := \{\langle m, n\rangle \ : \ m + n = k\}$$

to

$$C_k := (\widehat{k-2}, \widehat{k-1}] \cap \mathbb{N}.$$

Let $k$ be given.

(1) For $\langle m,n\rangle \in A_k = \{\langle 1,k-1\rangle, \langle 2,k-2\rangle, \cdots, \langle k-1,1\rangle\}$,

$$\begin{aligned}\widehat{k-2} = \widehat{m+n-2} < p(m,n) &= \widehat{m+n-2} + m \\ &= \widehat{k-2} + m \leq \widehat{k-2} + (k-1) = \widehat{k-1}\end{aligned}$$

This means $p$ maps $A_k$ into $C_k$.

(2) Let $\langle m_1,n_1\rangle$, $\langle m_2,n_2\rangle \in A_k$. Suppose that $p(\langle m_1,n_1\rangle) = p(\langle m_2,n_2\rangle)$, then

$$\widehat{k-2} + m_1 = \widehat{k-2} + m_2$$

So $m_1 = m_2$, and $n_1 = n_2$. This indicates, that $f$ is one-to-one from $A_k$ to $C_k$.

(3) Let $N \in C_k$. Denote

$$N = \widehat{k-2} + j$$

Choose $\langle m,n\rangle = \langle j,k-j\rangle$. Then $p(m,n) = N$.

Since $\mathbb{N}\times\mathbb{N} = \sqcup_{k=2}^{\infty} A_k$, $\mathbb{N} = \sqcup_{k=2}^{\infty}\Big((\widehat{k-2}, \widehat{k-2}] \cap \mathbb{N}\Big)$, which satisfies all conditions of previous example. Hence $p$ is bijective. □

Our next mission is that $\mathbb{Q}^+ \preceq \mathbb{N}\times\mathbb{N}$. Since we already have $\mathbb{N} \preceq \mathbb{Q}^+$, we'll show that $\mathbb{N} \approx \mathbb{Q}^+$.

**13.** $\mathbb{Q}^+ \preceq \mathbb{N}\times\mathbb{N}$ *because, we may choose* $f : \mathbb{Q}^+ \to \mathbb{N}\times\mathbb{N}$ *by*

$$f(\frac{q}{p}) = \langle q,p\rangle \qquad with\, gcd(p,q) = 1,$$

*i.e.*

$$f = \{\langle\ \frac{a}{b}, \langle a,b\rangle\ \rangle : a,b \in \mathbb{N},\ gcd(a,b) = 1\}$$

*i.e.*

$$\frac{n}{m} \mapsto \langle \frac{n}{gcd(n,m)}, \frac{m}{gcd(n,m)}\rangle \qquad for\, m,n \in \mathbb{N}.$$

*To show that $f$ is injective, given $\frac{m_1}{n_1}, \frac{m_2}{n_2} \in \mathbb{Q}^+$ such that $gcd(m_1,n_1) = gcd(m_2,n_2) = 1$ and $f(\frac{m_1}{n_1}) = f(\frac{m_2}{n_2})$. Since $\langle m_1,n_1\rangle = \langle m_2,n_2\rangle$, it follows that $m_1 = m_2$ and $n_1 = n_2$.*