14. *If $B \approx \mathbb{N}$, $A \subset B$, and $\mathbb{N} \preceq A$, then $A \approx \mathbb{N}$.*

Proof. Denote

$$B = \{b_1, b_2, \cdots\}$$

Then we hope to construct a sequence representing A, i.e. $A = \{y_1, y_2, \cdots\}$.

Let n_1 be the smallest integer n such that $b_n \in A$, and write $A_1 = A \setminus \{n_1\}$.

Let n_2 be the smallest integer $n>n_1$ such that $b_n \in A_1$, and write $A_2 = A_1 \setminus \{n_2\}$.

Let n_3 be the smallest integer $n>n_2$ such that $b_n \in A_2$, and write $A_3 = A_2 \setminus \{n_3\}$.

$\vdots$

The process can repeat permanently because $\mathbb{N} \preceq A_j$ for all $j \in \mathbb{N}$ (Why?). Since $A_j \neq \varnothing$, n_{j+1} can be chosen (from A_j). Next, to show

$$A = B_0 := \{b_{n_1}, b_{n_2}, \cdots\}$$

$B_0 \subset A$ is trivial. Let $a \in A \subset B$. Then $a = b_m$ for some $m \in \mathbb{N}$. Because $1 \leq n_1<n_2<\cdots$, we know $m \in (n_j, n_{j+1}]$. It follows that $a = b_{n_{n+1}}$. Hence $A \subset B_0$. Therefore $A = B_0$, and then $A \approx \mathbb{N}$. □

The following two propositions are prepared for the third proposition.

15. *If $A \preceq C$, then $A \approx B$ for some $B \subset C$. The reason is that, if f is one-to-one from A to C, then f is from A onto $B := f(A) \subset C$.*

16. *If $A \preceq B_1$ and $B_1 \approx B_2$, then $A \preceq B_2$. We derive this as follow: If f is one-to-one from A to B_1 and g is a bijection from B_1 to B_2, then $g \circ f$ is injective from A to B_2.*

17. $\mathbb{N} \approx \mathbb{Q}^+$.

Proof. We know that $\mathbb{Q}^+ \preceq \mathbb{N} \times \mathbb{N}$. Then $\mathbb{Q}^+ \approx N$ for some $N \subset \mathbb{N} \times \mathbb{N}$. The result is immediate from preceding examples. □

18. *We can now show equinumerosity between $\mathbb{Z}$ and $\mathbb{Q}$. Since we already have a one-to-one function f from $\mathbb{N}$ onto $\mathbb{Q}^+$. Let $F : \mathbb{Z} \to \mathbb{Q}$, by*

$$F(x) = \begin{cases} f(x), & \text{if } n \in \mathbb{N}; \\ 0, & \text{if } x = 0; \\ -f(-x), & \text{if } x \in -\mathbb{N}. \end{cases} \tag{3}$$

Then F is bijective.

We're going to show a more astonishing property, that $\mathbb{Q}$ contains (strictly) less elements than $\mathbb{R}$. Before reaching a precise proof, we need some preparation.

19. *If $A \approx B$, $B \approx C$, then $A \approx C$. A simple proof is: Suppose that f, g are bijections between A, B and C. Then $g \circ f$ is bijective from A to C.*

20. $\mathbb{R}^+ \approx \mathbb{R}$ *because, the functions* $f(x) := e^x$ *and*

$$g(x) = \begin{cases} x+1, & \textit{if } x \geq 0; \\ \frac{-x}{1-x}, & \textit{if } x<0. \end{cases} \tag{4}$$

are one-to-one from $\mathbb{R}$ *to* $\mathbb{R}^+$.

21. $\mathbb{N} \prec \mathbb{R}^+$.

Proof. It's clear that $\mathbb{N} \preceq \mathbb{R}^+$. Assume that $\mathbb{R}^+ = \{f_0, f_1, f_2, \cdots\}$, written in a sequence. We list the elements below in decimal expression.

$$\begin{aligned} f_0 &= f_{00}.f_{01}f_{02}f_{03}f_{04}\cdots \\ f_1 &= f_{10}.f_{11}f_{12}f_{13}f_{14}\cdots \\ f_2 &= f_{20}.f_{21}f_{22}f_{23}f_{24}\cdots \\ f_3 &= f_{30}.f_{31}f_{32}f_{33}f_{34}\cdots \\ &\vdots \end{aligned}$$

The main idea is to construct a decimal number which is none of the f_j's. Denote $f_\infty = f_{\infty 0}.f_{\infty 1}f_{\infty 2}f_{\infty 3}\cdots$, where

$$f_{\infty j} = \begin{cases} 7, & \text{if } f_{jj} = 3; \\ 3, & \text{Otherwise.} \end{cases} \tag{5}$$

Then $f_\infty \in \mathbb{R}^+$. It is some f_k, but $f_{\infty k} \neq f_{kk}$, a contradiction. Hence $\mathbb{N} \not\approx \mathbb{R}^+$. This means $\mathbb{N} \prec \mathbb{R}^+$. □

22. *We can now show that* $\mathbb{Q} \prec \mathbb{R}$. *It's well-known that* $\mathbb{Q} \preceq \mathbb{R}$. *Assume that* $\mathbb{Q} \approx \mathbb{R}$. *Since* $\mathbb{N} \approx \mathbb{Q}$ *and* $\mathbb{R} \approx \mathbb{R}^+$, *it follows that* $\mathbb{N} \approx \mathbb{R}^+$, *a contradiction.*

The last thing we hope to verify is that $\mathbb{R} \approx \mathbb{C}$. Similarly as before, for convenience, we divid it into several parts.

23. $(0,1) \approx (0,1) \times (0,1)$.

To give a precise proof, we need the help of Schröder-Bernstein Theorem, i.e. if $A \preceq B$ and $B \preceq A$ then $A \approx B$.

Proof. The map $x \mapsto (x, \frac{1}{2})$ is an injection. Define $f : (0,1) \times (0,1) \to (0,1)$ by

$$(a,b) := (0.a_1a_2a_3\cdots\ ,\ 0.b_1b_2b_3\cdots) \mapsto 0.a_1b_1a_2b_2a_3b_3\cdots$$

where none of a, b has consecutive 9's, i.e. neither a nor b is in the form $0.d_1d_2\cdots d_j9999\cdots$. Then if $a = 0.a_1a_2\cdots$, $b = 0.b_1b_2\cdots$, $c = 0.c_1c_2\cdots$ and $d = 0.d_1d_2\cdots$ are such that $f(a,b) = f(c,d)$, we write

$$0.a_1b_1a_2b_2\cdots = 0.c_1d_1c_2d_2\cdots$$

According to uniqueness of decimal expression, it follows that $a_j = c_j$ and $b_j = d_j$. Hence f is an injection. The property is then proved by Schröder-Bernstein Theorem. □

Note that the function f is not onto. In fact, we can find no pair mapping to the number $0.010191919191\cdots$ in $(0,1)$, hence we need to show *double injectivity*.

The following is a slight modification to our final goal.

24. *For* $(a,b) \subset \mathbb{R}$, $(a,b) \approx (0,1)$. *The map can be chosen as* $f(x) = (b-a)x + a$.

25. $(0,1) \approx \mathbb{R}$. *A bijection between them can be chosen as*

$$f(x) = \frac{x}{1+|x|}$$

from $\mathbb{R}$ *onto* $(-1,1)$ *or*

$$g(x) = \tan x$$

from $(-\frac{\pi}{2}, \frac{\pi}{2})$ *to* $\mathbb{R}$.

26. $(0,1) \times (0,1) \approx \mathbb{R} \times \mathbb{R}$.

Proof. Let $f : (0,1) \to \mathbb{R}$ be bijective. Then

$$F(a,b) = (f(a), f(b)) \quad on\, (0,1) \times (0,1)$$

is a bijection that we desire. □

27. *Since* $\mathbb{R} \approx (0,1)$, $(0,1) \approx (0,1)\times(0,1)$, $(0,1)\times(0,1) \approx \mathbb{R}\times\mathbb{R}$, *we conclude that* $\mathbb{R} \approx \mathbb{R}\times\mathbb{R}$.

28. $\mathbb{R} \times \mathbb{R} \approx \mathbb{C}$, *because we have the function* $k(x,y) = x + iy$. *Therefore we obtain the final equinumerosity*

$$\mathbb{R} \approx \mathbb{C}.$$

Although we've shown that the five famous sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ have only 2 "levels" about numbers of elements, yet we even have no idea about what an infinite set really is. Our intuition might tell us that a finite set must contain some n elements. That means, it must be equinumerous to some *standard set* which we consider to have n elements. In the following discussion, this standard set will naturally be chosen as $I_n := \{1, 2, 3, \cdots, n\}$.

Except for the notion of an infinite set, we have another notion called countable sets. The difference from countability depends on whether the given set can be listed in a (finite of infinite) sequence.

29. *We now make a convention that* $\mathbb{W}$ *means the whole number set. i.e.* $\mathbb{W} = \mathbb{N} \cup \{0\}$. *Define* $I_n := \{1, 2, \cdots, n\} = \{k \in \mathbb{N} : 1 \le k \le n\}$, *and* $I_0 = \varnothing$, *for convenience of later consideration.*

30. *A set S is called finite if $S \approx I_n$ for some $n \in \mathbb{W}$. If S is not finite, then S is called infinite. Note that the "new" notion leads to no contradiction about what are discussed before because we did not mention the word "infinite" previously.*

31. *A set K is called countable if all elements of K are in a (finite or infinite) sequence, i.e. if $K \approx I_n$ for some $n \in \mathbb{W}$ or $K \approx \mathbb{N}$. In the later case, K is also called denumerable. So K is countable if K is either finite or denumerable. If K is not countable, we say K is uncountable.*

There are many "trivial" statements concerning infinite sets, but not all of them is easy to prove.

32. *If S is an infinite set, and $T \subset S$, then either T or $S \setminus T$ is infinite.*

Proof. Assume that S and $S \setminus T$ are both finite. Let $f : I_n \to S$ and $g : I_m \to S \setminus T$ be bijections. Then define $\varphi : I_{m+n} \to S$ as follow:

$$\varphi(x) = \begin{cases} f(x), & \text{if } x = 1, 2, 3 \cdots, n; \\ g(x-n), & \text{if } x = n+1, n+2, \cdots, n+m. \end{cases} \tag{6}$$

Then φ is one-to-one from I_{n+m} onto S, a contradiction to the fact that S is infinite. □

33. *The union of a finite set and a denumerable set is denumerable because, if the finite set is expressed as $\{a_1, a_2, \cdots, a_n\}$ while the other $\{b_1, b_2, \cdots\}$, then the union $\{a_1, \cdots, a_n, b_1, b_2, \cdots\}$ is equinumerous to a subset (why not exactly ?) of $\{\langle a, 1\rangle, \cdots \langle a, n\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \cdots\}$. Moreover, $\mathbb{N} \preceq$ the union, we conclude that $\mathbb{N}$ is equinumerous to the union.*

34. *The disjoint union of two denumerable sets are denumerable. Therefore, the disjoint union of n denumerable sets are also denumerable.*

Proof. It easy to check that $\mathbb{N}$ is equinumerous to $2\mathbb{N}$ and $2\mathbb{N}+1$ (We define $2\mathbb{N} = \{2, 4, 6, 8, \cdots\}$ and $2\mathbb{N}+1 = \{1, 3, 5, \cdots\}$.). Let A, B be two disjoint sets. Choose $f : 2\mathbb{N} \to A$, $g : 2\mathbb{N}+1 \to B$ as bijections. Define $G : \mathbb{N} \to A \sqcup B$ by

$$G(x) = \begin{cases} f(x), & \text{if } x \text{ is even;} \\ g(x), & \text{if } x \text{ is odd.} \end{cases} \tag{7}$$

Then G shows that $A \sqcup B$ is denumerable. □

35. *How about the union of n denumerable sets?*
How about a denumerable union of denumerable sets?

For the second assertion, we need some observation.

36. *The denumerable union of pairwise disjoint denumerable sets are denumerable.*

Proof. Denote these sets by $S_1, S_2, \cdots$, where

$$\begin{aligned} S_1 &= \{s_{11}, s_{12}, s_{13}, \cdots\} \\ S_2 &= \{s_{21}, s_{22}, s_{23}, \cdots\} \\ S_3 &= \{s_{31}, s_{32}, s_{33}, \cdots\} \\ &\vdots \end{aligned}$$

are their elements. Then the mapping $\langle i, j\rangle \leftrightarrow s_{ij}$ is bijective. Hence $\mathbb{N} \approx \bigcup_{n\in\mathbb{N}} S_n$. □

37. *A set S is countable if and only if $S \preceq \mathbb{N}$.*

Proof. If S is denumerable, then $S \approx \mathbb{N}$ and hence $S \preceq \mathbb{N}$. If S if finite, then $S \preceq \mathbb{N}$ because $I_n \preceq \mathbb{N}$. Conversly, Let f be one-to-one from S into $\mathbb{N}$. Let $T = Im(f)$. Then

(a) If T contains no maximum, then $\mathbb{N} \preceq T$ (why?). Since $T \subset \mathbb{N}$, we obtain that $\mathbb{N} \approx T$. Since f is onto T, it follows that $S \approx T$.

(b) If T contains a maximum, say $n_0 \in \mathbb{N}$. Then $T \subset I_{n_0}$. Hence T is equinumerous to some I_m (why? This might be proved by induction.). Similarly, the assertion is proved by $S \approx T \approx I_m$.

□

38. *The denumerable union of pairwise disjoint countable sets are countable.*