Where to go ??

The Essense of Schröder-Bernstein Theorem

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A significant question is that: Whether when $A \prec B$ and $B \prec A$, it is necessary that $A \approx B$?? Before handling this, we firstly consider a practical example.

Example 1. *To consider whether* $[0,1] \approx [0,1)$[1]*, we ask if there's a bijective* $f : [0,1] \to [0,1)$*. Suppose it exists, who is the images of* 1*? It may be chosen, say* $f(1) = \frac{1}{2}$*, and then, since* f *should be injective, who is* $f(\frac{1}{2})$ *? If we set for example that* $f(\frac{1}{2}) = \frac{1}{4}$*, we still have define* $f(\frac{1}{4}), f(\frac{1}{8})$ *and so on. Now, the idea emerges.*

Define

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \{\frac{1}{2^k} : k = 0,1,2,\cdots\}; \\ x, & \text{otherwise.} \end{cases} \tag{1}$$

We will confirm ourselves their equinumerousity.

Example 2 (Schröder-Bernstein Theorem)**.** *If* $A \prec B$ *and* $B \prec A$*, then* $A \approx B$*.*

Draw two lines, by which we represent A *and* B*. Then pick up an injection* f *from* A *onto* $f(A) \subset B$ *(We assume* $f(A) \neq B$*). The key point is that we will pretend* $f(A)$ *and* A *are the same. (At least we say they are closely related, indistinguishable, or essentially the same.)*

Now we only focus on B *because* A *is viewed as a subset of* B*. Let*

$$B_0 = B \setminus f(A).$$

The one-to-one function $h : B \to A$ *is now viewed from* B *onto the "subset"* A*, as in the previous example that* $\frac{x}{2}$ *sends* $[0,1]$ *injectively into* $[0,1)$*. The where to go question occurs on the function* h *and we are looking for it. So we want to answer where* B_0*(In Example 1 case, it is* $= [0,1] \setminus [0,1) = \{1\}$*) is sent, and who plays a role as* $x/2$ *in Example 1?*

But don't forget we already have such an one-to-one function $h : B \to A$ *by condition, so everything is well-prepared and we should focus on a precise proof. From now on, since we are going to operate* f *and* h*, we should write down the embedding function* $f : A \to B$ *when we "iterally apply"* h *on* B_0*.*

Define

$$\begin{aligned} B_1 &= f(h(B_0)) \\ B_2 &= f(h(B_1)) \\ &\vdots \end{aligned}$$

[1] This is changed from previous version that there is only one added point

Then the following is a bijection g from B onto A. Choose

$$g(x) = \begin{cases} h(x), & \text{if } x \in B_j; \\ f^{-1}(x), & \text{otherwise.} \end{cases} \tag{2}$$

Then all above is the same as Example 1.

Example 3. *This is another theorem about the method: If $X \supset Y \supset X_1$, and $X \approx X_1$, then $X \approx Y$.*

The way we solve it is like this: Since we have an f, one-to-one from X onto X_1, yet $f(Y) \subset X_1$ (Assume that $f(Y) \neq X_1$), denoted by Y_1. It's clear that f is one-to-one from Y onto Y_1, so $X_2 := f(X_1)$ is a proper subset of Y_1. Continually repeat this process, we obtain the family of sets

$$X = X_0 \supset Y = Y_0 \supset X_1 \supset Y_1 \supset X_2 \supset Y_2 \supset \cdots$$

*All we perform above are natural. We might now start our "where-to-go" method. Eventually we find an F from X to Y (Note that F is **from X to Y**.), by*

$$F(x) = \begin{cases} f(x), & \text{if } x \in X_j \setminus Y_j; \\ x, & \text{otherwise.} \end{cases} \tag{3}$$

Finally, we ought to show that $F(x)$ indeed works.

An important remark is:

Example 4. *The last two examples are equivalent.*