Where to go ??

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A significant question is that: Whether when $A \prec B$ and $B \prec A$, it is necessary that $A \approx B$?? Before handling this, we firstly consider a practical example.

Example 1. *To consider the two sets:* $[-1,1]$ *and* $(-1,1)$. *To ask whether there's a bijective* $f : [-1,1] \to (-1,1)$*? If it exists, who are the images of* $-\frac{1}{2}$ *and* $\frac{1}{2}$*? It may be chosen, say,* $f(-1) = \frac{1}{2}$, *and* $f(1) = \frac{1}{2}$, *and then, since* f *is convented to be one-to-one, so where will* $f(-\frac{1}{2})$ *and* $f(\frac{1}{2})$ *go afterward? We might imagine, for example,* $f(-\frac{1}{2}) = -\frac{1}{4}$ *and* $f(\frac{1}{2}) = \frac{1}{4}$. *Now, the idea emerges.*

Define

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \{\pm\frac{1}{2^k} : k = 0, 1, 2, \cdots\}; \\ x, & \text{otherwise.} \end{cases} \tag{1}$$

We will confirm ourselves their equinumerousity.

Example 2 (Schröder-Bernstein Theorem)**.** *If* $A \prec B$ *and* $B \prec A$, *then* $A \approx B$.

Draw two lines, by which we hope to represent the sets A *and* B. *Then we pick up an injection* f *from* A *onto* $f(A) \subset B$ *(We assume that* $f(A) \neq B$*). The key point we keep in mind is that we will pretend that* $f(A)$ *and* A *itself are "the same". (At least wemay say that they are closely related, or almost the same.)*

Now we only focus on B. *Let*

$$B_0 = B \setminus f(A)$$

Imagine that g *is such an one-to-one function from* B *onto* A, *which we "view" as a function from* B *onto the part* $f(A)$, *of* B. *(Remeber we always "tell" , meanwhile we have "convinced" ourselves that* A *and* $f(A)$ *are indistinguishable.) Then we will soon ask where* B_0 *does map to? It is* $g(B_0)$, *which we "agreed" to "be"* $f(g(B_0))$, *which we call* B_1. *So where will* B_1 *map to?*

We denote B_2 *for* $f(g(B_1))$, B_3 *for* $f(g(B_2))$, *...*

What is remarkable is that, from B *to* A, g *plays a role as an injection. This inspires us that if we have a injection from* B *to* A *then everything is done. Fortunately we do, by condition!*

Let h be one-to-one from B to A. Define

$$\begin{aligned} B_1 &= f(h(B_0)) \\ B_2 &= f(h(B_1)) \\ &\vdots \end{aligned}$$

Then the following is a bijection g from B onto A. Choose

$$g(x) = \begin{cases} h(x), & \text{if } x \in B_j; \\ f^{-1}(x), & \text{otherwise.} \end{cases} \tag{2}$$

Note that we have to distinguish $f(A)$ from A itself when we come to a precise proof. So above is a necessary modification, and the last step is to show bijectiveness.

Example 3. *This is another theorem about the method: If $X \supset Y \supset X_1$, and $X \approx X_1$, then $X \approx Y$.*

The way we solve it is like this: Since we have an f, one-to-one from X onto X_1, yet $f(Y) \subset X_1$ (Assume that $f(Y) \neq X_1$), denoted by Y_1. It's clear that f is one-to-one from Y onto Y_1, so $X_2 := f(X_1)$ is a proper subset of Y_1. Continually repeat this process, we obtain the family of sets

$$X = X_0 \supset Y = Y_0 \supset X_1 \supset Y_1 \supset X_2 \supset Y_2 \supset \cdots$$

*All we perform above are natural. We might now start our "where-to-go" method. Eventually we find an F from X to Y (Note that F is **from X to Y**.), by*

$$F(x) = \begin{cases} f(x), & \text{if } x \in X_j \setminus Y_j; \\ x, & \text{otherwise.} \end{cases} \tag{3}$$

Finally, we ought to show that $F(x)$ indeed works.

An important remark is:

Example 4. *The last two examples are equivalent.*