Exercise for Chapter 3

April 5, 2012

1. How many elements does the set $\{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\}$ have ? How about subsets ?

2. Show that if $A \cap B = \varnothing$ and $A \cup B = \mathbb{R}$, then for each $t \in \mathbb{R}$, we have either (I) $t \in A$, $t \notin B$, or (II) $t \in B$, $t \notin A$.

3. (a) Show that $T \times \cup_{\alpha\in\Lambda} S_\alpha = \cup_{\alpha\in\Lambda} T \times S_\alpha$.
   (b) Show that $\wp(\cap_\alpha S_\alpha) = \cap_\alpha(\wp S_\alpha)$.
   (c) Show that $\bigcup_{\alpha\in\Lambda} X \setminus S_\alpha = X \setminus \bigcap_{\alpha\in\Lambda} S_\alpha$.

4. Show that if A is a subset of all sets, then $A = \varnothing$.

5. (a) Show that $\wp(A \cup B) \supseteq \wp A \cup \wp B$??
   (b) Show that the equality holds if and only if $A \subset B$ or $B \subset A$.

6. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive real numbers. Show that $\lim_{n\to\infty} a_n = 0$ if and only if $\cap_{j=1}^{\infty}[0, a_j) = \{0\}$.

7. Let $\wp X$ denote the power set of X.
   (a) [15%] Show that if $A \subset B$ then $\wp A \subset \wp B$.
   (b) [10%] Show that $\{\varnothing, \{\varnothing\}\} \in \wp\wp\wp S$ for any set S. [Hint: Who is the subset of all sets ??]

8. A "good set" is defined as a nonempty proper subset of $\mathbb{Q}^+$. i.e. A is a good if and only if $\varnothing \subsetneq A \subsetneq \mathbb{Q}^+$.

   The operations of two good sets are defined by

$$A + B := \{X + Y : x \in A,\ Y \in B\}$$
$$A \cdot B := \{X \cdot Y : x \in A,\ Y \in B\}$$

   Show that for good sets ξ, η, ζ, we always have $\xi \cdot (\eta + \zeta) = \xi \cdot \eta + \xi \cdot \zeta$.

9. (a) For a, b, show that if $\{a, b\} = \{c, d\}$ then we have either (I) $a = c$ and $b = d$, or (II) $a = d$ and $b = c$.
   (b) Define $\langle a, b\rangle = \{\{\text{a}\},\{\text{a,b}\}\}$. Show that if $\langle a, b\rangle = \langle c, d\rangle$ then $a = c$ and $b = d$.