Multiple Table

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With the help of Addition Lemma, we're going to construct the multiple lemma. An important fact is that,

$$\overline{a_1 a_0} + \overline{b_1 b_0} = \overline{c_1 c_0},$$

where, whenever $a_1 + b_1 \leq 8$, if we denote $a_0 + b_0 = m \cdot \mathbf{T} + n$, $c_0 = n$ and $c_1 = a_1 + b_1 + m$.

Proof.

$\overline{a_1 a_0} + \overline{b_1 b_0} = (a_1 \cdot \mathbf{T} + a_0) + (b_1 \cdot \mathbf{T} + b_0) = ((a_1 \cdot \mathbf{T} + a_0) + b_1 \cdot \mathbf{T}) + b_0$
$= (a_1 \cdot \mathbf{T} + (a_0 + b_1 \cdot \mathbf{T})) + b_0 = (a_1 \cdot \mathbf{T} + (b_1 \cdot \mathbf{T} + a_0)) + b_0 = ((a_1 \cdot \mathbf{T} + b_1 \cdot \mathbf{T}) + a_0) + b_0$
$= (a_1 \cdot \mathbf{T} + b_1 \cdot \mathbf{T}) + (a_0 + b_0) = (a_1 + b_1) \cdot \mathbf{T} + (a_0 + b_0).$

Note that $a_0 + b_0 \leq 9 + 9 = 18$. If $a_0 + b_0 > 10$, then $a_0 + b_0 = 1 \cdot \mathbf{T} + c_0 = m \cdot T + c_0$ for some c_0 and $m = 1$. At this time,

$$\overline{a_1 a_0} + \overline{b_1 b_0} = (a_1 + b_1) \cdot \mathbf{T} + m \cdot \mathbf{T} + c_0 = (a_1 + b_1 + m) \cdot \mathbf{T} + c_0.$$

If $a_0 + b_0 < 10$, then $a_0 + b_0 = 0 \cdot \mathbf{T} + c_0 = m \cdot \mathbf{T} + c_0$, where $m = 0$ and $c_0 = a_0 + b_0$.
Write $c_1 = a_1 + b_1 + m$. Then $c_1 \leq 8 + 1 = 9$. Hence $\overline{a_1 a_0} + \overline{b_1 b_0} = \overline{c_1 c_0}$. $\qquad\square$

LEMMA.

$1 \cdot 1 = 1$	$1 \cdot 2 = 2$	$1 \cdot 3 = 3$	$1 \cdot 4 = 4$	$1 \cdot 5 = 5$
$2 \cdot 1 = 2$	$2 \cdot 2 = 4$	$2 \cdot 3 = 6$	$2 \cdot 4 = 8$	$2 \cdot 5 = 10$
$3 \cdot 1 = 3$	$3 \cdot 2 = 6$	$3 \cdot 3 = 9$	$3 \cdot 4 = 12$	$3 \cdot 5 = 15$
$4 \cdot 1 = 4$	$4 \cdot 2 = 8$	$4 \cdot 3 = 12$	$4 \cdot 4 = 16$	$4 \cdot 5 = 20$
$5 \cdot 1 = 5$	$5 \cdot 2 = 10$	$5 \cdot 3 = 15$	$5 \cdot 4 = 20$	$5 \cdot 5 = 25$
$6 \cdot 1 = 6$	$6 \cdot 2 = 12$	$6 \cdot 3 = 18$	$6 \cdot 4 = 24$	$6 \cdot 5 = 30$
$7 \cdot 1 = 7$	$7 \cdot 2 = 14$	$7 \cdot 3 = 21$	$7 \cdot 4 = 28$	$7 \cdot 5 = 35$
$8 \cdot 1 = 8$	$8 \cdot 2 = 16$	$8 \cdot 3 = 24$	$8 \cdot 4 = 32$	$8 \cdot 5 = 40$
$9 \cdot 1 = 9$	$9 \cdot 2 = 18$	$9 \cdot 3 = 27$	$9 \cdot 4 = 36$	$9 \cdot 5 = 45$

$1 \cdot 6 = 6$	$1 \cdot 7 = 7$	$1 \cdot 8 = 8$	$1 \cdot 9 = 9$
$2 \cdot 6 = 12$	$2 \cdot 7 = 14$	$2 \cdot 8 = 16$	$2 \cdot 9 = 18$
$3 \cdot 6 = 18$	$3 \cdot 7 = 21$	$3 \cdot 8 = 24$	$3 \cdot 9 = 27$
$4 \cdot 6 = 24$	$4 \cdot 7 = 28$	$4 \cdot 8 = 32$	$4 \cdot 9 = 36$
$5 \cdot 6 = 30$	$5 \cdot 7 = 35$	$5 \cdot 8 = 40$	$5 \cdot 9 = 45$
$6 \cdot 6 = 36$	$6 \cdot 7 = 42$	$6 \cdot 8 = 48$	$6 \cdot 9 = 54$
$7 \cdot 6 = 42$	$7 \cdot 7 = 49$	$7 \cdot 8 = 56$	$7 \cdot 9 = 63$
$8 \cdot 6 = 48$	$8 \cdot 7 = 56$	$8 \cdot 8 = 64$	$8 \cdot 9 = 72$
$9 \cdot 6 = 54$	$9 \cdot 7 = 63$	$9 \cdot 8 = 72$	$9 \cdot 9 = 81$

Proof. By definition frequently, $1 \cdot 1 = 1$, $1 \cdot 2 = 1 \cdot 1' = 1 \cdot 1 + 1 = 1 + 1 = 2$, $1 \cdot 3 = 1 \cdot 2' = 1 \cdot 2 + 1 = 2 + 1 = 3$, $1 \cdot 4 = 1 \cdot 3' = 1 \cdot 3 + 1 = 3 + 1 = 4$, $1 \cdot 5 = 1 \cdot 4' = 1 \cdot 4 + 1 = 4 + 1 = 5$, $1 \cdot 6 = 1 \cdot 5' = 1 \cdot 5 + 1 = 5 + 1 = 6$, $1 \cdot 7 = 1 \cdot 6' = 1 \cdot 6 + 1 = 6 + 1 = 7$, $1 \cdot 8 = 1 \cdot 7' = 1 \cdot 7 + 1 = 7 + 1 = 8$, $1 \cdot 9 = 1 \cdot 8' = 1 \cdot 8 + 1 = 8 + 1 = 9$. Next, $2 \cdot 1 = 2$, $2 \cdot 2 = 2 \cdot 1' = 2 \cdot 1 + 2 = 2 + 2 = 4$, $2 \cdot 3 = 2 \cdot 2' = 2 \cdot 2 + 2 = 4 + 2 = 6$, $2 \cdot 4 = 2 \cdot 3' = 2 \cdot 3 + 2 = 6 + 2 = 8$, $2 \cdot 5 = 2 \cdot 4' = 2 \cdot 4 + 2 = 8 + 2 = 10$, $2 \cdot 6 = 2 \cdot 5' = 2 \cdot 5 + 2 = 10 + 2 = 12$, $2 \cdot 7 = 2 \cdot 6' = 2 \cdot 6 + 2 = 12 + 2 = 14$, $2 \cdot 8 = 2 \cdot 7' = 2 \cdot 7 + 2 = 14 + 2 = 16$, $2 \cdot 9 = 2 \cdot 8' = 2 \cdot 8 + 2 = 16 + 2 = 18$.

Let's proceed. $3 \cdot 1 = 3$. $3 \cdot 2 = 3 \cdot 1' = 3 \cdot 1 + 3 = 3 + 3 = 6$, $3 \cdot 3 = 3 \cdot 2' = 3 \cdot 2 + 3 = 6 + 3 = 9$, $3 \cdot 4 = 3 \cdot 3' = 3 \cdot 3 + 3 = 9 + 3 = 12$, $3 \cdot 5 = 3 \cdot 4' = 3 \cdot 4 + 3 = 12 + 3 = 15$, $3 \cdot 6 = 3 \cdot 5' = 3 \cdot 5 + 3 = 15 + 3 = 18$, $3 \cdot 7 = 3 \cdot 6' = 3 \cdot 6 + 3 = 18 + 3 = 21$, $3 \cdot 8 = 3 \cdot 7' = 3 \cdot 7 + 3 = 21 + 3 = 24$, $3 \cdot 9 = 3 \cdot 8' = 3 \cdot 8 + 3 = 24 + 3 = 27$. $4 \cdot 1 = 4$. $4 \cdot 2 = 4 \cdot 1' = 4 \cdot 1 + 4 = 4 + 4 = 8$, $4 \cdot 3 = 4 \cdot 2' = 4 \cdot 2 + 4 = 8 + 4 = 12$, $4 \cdot 4 = 4 \cdot 3' = 4 \cdot 3 + 4 = 12 + 4 = 16$, $4 \cdot 5 = 4 \cdot 4' = 4 \cdot 4 + 4 = 16 + 4 = 20$, $4 \cdot 6 = 4 \cdot 5' = 4 \cdot 5 + 4 = 20 + 4 = 24$, $4 \cdot 7 = 4 \cdot 6' = 4 \cdot 6 + 4 = 24 + 4 = 28$, $4 \cdot 8 = 4 \cdot 7' = 4 \cdot 7 + 4 = 28 + 4 = 32$, $4 \cdot 9 = 4 \cdot 8' = 4 \cdot 8 + 4 = 32 + 4 = 36$. $5 \cdot 1 = 5$. $5 \cdot 2 = 5 \cdot 1' = 5 \cdot 1 + 5 = 5 + 5 = 10$, $5 \cdot 3 = 5 \cdot 2' = 5 \cdot 2 + 5 = 10 + 5 = 15$, $5 \cdot 4 = 5 \cdot 3' = 5 \cdot 3 + 5 = 15 + 5 = 20$, $5 \cdot 5 = 5 \cdot 4' = 5 \cdot 4 + 5 = 20 + 5 = 25$, $5 \cdot 6 = 5 \cdot 5' = 5 \cdot 5 + 5 = 25 + 5 = 30$, $5 \cdot 7 = 5 \cdot 6' = 5 \cdot 6 + 5 = 30 + 5 = 35$, $5 \cdot 8 = 5 \cdot 7' = 5 \cdot 7 + 5 = 35 + 5 = 40$, $5 \cdot 9 = 5 \cdot 8' = 5 \cdot 8 + 5 = 40 + 5 = 45$. $6 \cdot 1 = 6$, $6 \cdot 2 = 6 \cdot 1' = 6 \cdot 1 + 6 = 6 + 6 = 12$, $6 \cdot 3 = 6 \cdot 2' = 6 \cdot 2 + 6 = 12 + 6 = 18$, $6 \cdot 4 = 6 \cdot 3' = 6 \cdot 3 + 6 = 18 + 6 = 24$, $6 \cdot 5 = 6 \cdot 4' = 6 \cdot 4 + 6 = 24 + 6 = 30$, $6 \cdot 6 = 6 \cdot 5' = 6 \cdot 5 + 6 = 30 + 6 = 36$, $6 \cdot 7 = 6 \cdot 6' = 6 \cdot 6 + 6 = 36 + 6 = 42$, $6 \cdot 8 = 6 \cdot 7' = 6 \cdot 7 + 6 = 42 + 6 = 48$, $6 \cdot 9 = 6 \cdot 8' = 6 \cdot 8 + 6 = 48 + 6 = 54$.

Moreover, $7 \cdot 1 = 7$, $7 \cdot 2 = 7 \cdot 1' = 7 \cdot 1 + 7 = 7 + 7 = 14$, $7 \cdot 3 = 7 \cdot 2' = 7 \cdot 2 + 7 = 14 + 7 = 21$, $7 \cdot 4 = 7 \cdot 3' = 7 \cdot 3 + 7 = 21 + 7 = 28$, $7 \cdot 5 = 7 \cdot 4' = 7 \cdot 4 + 7 = 28 + 7 = 35$, $7 \cdot 6 = 7 \cdot 5' = 7 \cdot 5 + 7 = 35 + 7 = 42$, $7 \cdot 7 = 7 \cdot 6' = 7 \cdot 6 + 7 = 42 + 7 = 49$, $7 \cdot 8 = 7 \cdot 7' = 7 \cdot 7 + 7 = 49 + 7 = 56$, $7 \cdot 9 = 7 \cdot 8' = 7 \cdot 8 + 7 = 56 + 7 = 63$. $8 \cdot 1 = 8$, $8 \cdot 2 = 8 \cdot 1' = 8 \cdot 1 + 8 = 8 + 8 = 16$, $8 \cdot 3 = 8 \cdot 2' = 8 \cdot 2 + 8 = 16 + 8 = 24$, $8 \cdot 4 = 8 \cdot 3' = 8 \cdot 3 + 8 = 24 + 8 = 32$, $8 \cdot 5 = 8 \cdot 4' = 8 \cdot 4 + 8 = 32 + 8 = 40$, $8 \cdot 6 = 8 \cdot 5' = 8 \cdot 5 + 8 = 40 + 8 = 48$, $8 \cdot 7 = 8 \cdot 6' = 8 \cdot 6 + 8 = 48 + 8 = 56$, $8 \cdot 8 = 8 \cdot 7' = 8 \cdot 7 + 8 = 56 + 8 = 64$, $8 \cdot 9 = 8 \cdot 8' = 8 \cdot 8 + 8 = 64 + 8 = 72$.

Finally, $9 \cdot 1 = 9$, $9 \cdot 2 = 9 \cdot 1' = 9 \cdot 1 + 9 = 9 + 9 = 18$, $9 \cdot 3 = 9 \cdot 2' = 9 \cdot 2 + 9 = 18 + 9 = 27$, $9 \cdot 4 = 9 \cdot 3' = 9 \cdot 3 + 9 = 27 + 9 = 36$, $9 \cdot 5 = 9 \cdot 4' = 9 \cdot 4 + 9 = 36 + 9 = 45$, $9 \cdot 6 = 9 \cdot 5' = 9 \cdot 5 + 9 = 45 + 9 = 54$, $9 \cdot 7 = 9 \cdot 6' = 9 \cdot 6 + 9 = 54 + 9 = 63$, $9 \cdot 8 = 9 \cdot 7' = 9 \cdot 7 + 9 = 63 + 9 = 72$, $9 \cdot 9 = 9 \cdot 8' = 9 \cdot 8 + 9 = 72 + 9 = 81$.

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