Exercise

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Example 1. *The set* $\{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\}$ *contains only 3 elements. So it has* 2^3 *subsets.*

Example 2. *Two sets* X *and* Y *are called identified, denoted by* $X = Y$*, provided that for each* $x \in X$*,* $x \in Y$*, and for each* $y \in Y$*,* $y \in X$*. Use it to show that whenever* $\{a, b\} = \{c, d\}$*, we must have either* $a = c, b = d$*, or,* $a = d, b = c$*.*

Proof. Divid the statement according to whether $a = b$ and $c = d$, into 4 cases, we might step by step get the conclusion. $\square$

Example 3. $A = B$ *if and only if* $\wp A = \wp B$*.*

Proof. Assume that $A = B$, we want to show $\wp A = \wp B$. Let $K \in \wp A$. Then $K \subset A$. Since $A \subset B$, yet we have $K \subset A \subset B$. We get $K \subset B$, i.e. $K \in \wp B$. Hence $\wp A \subset \wp B$. Similarly we have $\wp B \subset \wp A$. Therefore $\wp A = \wp B$.

Conversely, assume that $\wp A = \wp B$, we ought to show that $A = B$. Let $x \in A$. Then $\{x\} \subset A$, and we obtain $\{x\} \in \wp A$. Because $\wp A = \wp B$, $\{x\} \in \wp B$. Hence $\{x\} \subset B$, so we get that $x \in B$. Hence $A \subset B$. Similarly we have $B \subset A$. Therefore $A = B$. $\square$

Example 4. $T \times (\cup_{a \in A} S_a) = \cup_{a \in A} T \times S_a$.

Proof. Let $x \in T \times (\cup_{a \in A} S_a)$. Then $x = (t, s)$ for some $t \in T, s \in \cup_{a \in A} S_a$. Accordingly we know that $s \in S_{a_0}$ for some $a_0 \in A$. Hence the ordered pair (t, s) is such that $t \in T$ and $s \in S_{a_0}$, and then $(t, s) \in T \times S_{a_0}$. Then we find that $x \in \cup_{a \in A} T \times S_a$. Hence $T \times (\cup_{a \in A} S_a) \subset \cup_{a \in A} T \times S_a$.

Conversely, given $y \in \cup_{a \in A} T \times S_a$, then $y \in T \times S_{a_0}$ for some $a_0 \in A$. We can write $y = (t, s)$ for some $t \in T$ and $s \in S_{a_0}$. It follows that $s \in \cup_{a \in A} S_a$, and $y = (t, s)$ is in fact in $T \times \cup_{a \in A} S_a$. Hence $T \times (\cup_{a \in A} S_a) \supset \cup_{a \in A} T \times S_a$. Therefore, $T \times (\cup_{a \in A} S_a) = \cup_{a \in A} T \times S_a$. $\square$

Example 5. $\cap_{j=1}^{\infty} [0, \frac{1}{j}) = \{0\}$.

Proof. Let $x \in \cap_{j=1}^{\infty} [0, \frac{1}{j})$. We ought to show that $x \in \{0\}$. i.e. $x = 0$. Assume not. If $x<0$, then $x \notin [0, \frac{1}{1})$, which is not permitted. Else if $x>0$, then we choose an $N \in \mathbb{N}$ such that $\frac{1}{N}<x$. Since x is in the intersection, x must be in $[0, \frac{1}{N})$, a contradiction. Therefore, $x = 0$, and $\cap_{j=1}^{\infty} [0, \frac{1}{j}) \subset \{0\}$.

Conversely, given $y \in \{0\}$, then $y = 0$. Since we know that $0 \leq 0<\frac{1}{j}$ for each $j \in \mathbb{N}$, $y \in \cap_{j=1}^{\infty} [0, \frac{1}{j})$. Hence $\cap_{j=1}^{\infty} [0, \frac{1}{j}) \supset \{0\}$. Therefore $\cap_{j=1}^{\infty} [0, \frac{1}{j}) = \{0\}$. $\square$

Example 6. *Let* c_n *be a strictly decreasing sequence of positive real numbers. If* $\lim_{n \to \infty} c_n = 0$ *then* $\cap_{j=1}^{\infty} [0, c_n) = \{0\}$*.*

Proof. Let $x \in \cap_{j=1}^{\infty} [0, c_n)$. We have to show that $x = 0$. It is clear that $x \geq 0$. It remains to show that $x>0$ would never occur. Assume that $x>0$, then because of the limit of the sequence there is an $N \in \mathbb{N}$ such that for each $n \geq N$, we have $c_n<x$. For this N, we can observe that $x>c_N$, but according to the intersection, $x \in [0, c_N)$, a contradiction. Hence $x = 0$.

The converse direction follows from the last example, which is not hard to prove. Hence $\cap_{j=1}^{\infty} [0, c_n) = \{0\}$. $\square$

Example 7. *The way to negate a statement is to exchange the quantifiers, and then negate the pattern forms. If we hope to negate that "for each $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, if $n>N$ then $|a_n - a|<\varepsilon$", we will write down the result: "There is an $\varepsilon_0>0$ such that for each $N \in \mathbb{N}$, there is an $n_N \in \mathbb{N}$, such that $n_N>N$ but $|a_n - a| \geq \varepsilon_0$ ".*

Example 8. *A way to express an ordered pair by means of an unordered pair is that: we may define $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. We ought to show that if $\langle a, b\rangle = \langle c, d\rangle$, then $a = c$ and $b = d$.*

Proof. By *example 3* we find that either (I) $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$ or (II) $\{a\} = \{c, d\}$ and $\{a, b\} = \{c\}$.

In case (II) we obtain that $c = d = a$ and $c = d = b$ because we know that $c \in \{c, d\}$ and c will consequencely belong to $\{a\}$, so that $a = c$ and similarly $a = d$. The later case is alike. Hence $a = c = b = d$.

In case (I), we know by $\{a\} = \{c\}$ that $a = c$. we firstly assume that $a = b$, then similarly we have $a = c = b = d$. If $a \neq b$, we hope to show that $c \neq d$. Aussume the contrary, that $c = d$, then the same case that $a = b = c = d$ would again occur, a contradiction. So $c \neq d$. Since $b \in \{a, b\}$, $b \in \{c, d\}$. If $b = c$ then $a = b = c$, which is excluded, so $b = d$. □

Example 9. *Let $x \geq 0$, $n \in \mathbb{N}$. Show that*

$$[x] + [x + \frac{1}{n}] + \cdots + [x + \frac{n-1}{n}] = [nx]$$

where [x] denotes the Gaussian Notation.

Proof. Define $I_k = [\frac{k-1}{n}, \frac{k}{n})$. Then for each $x \geq 0$, there is a k such that $x \in I_k$. We hope to perform induction on k. i.e. When $x \in I_k$, the equality holds.

(1) If $k = 1$, then $0 \leq x<\frac{1}{n}$. We find that $0 \leq x + \frac{j}{n}<x + \frac{n-1}{n} = 1$ for $j = 0, 1, 2, \cdots, n-1$. So that $[x + \frac{j}{n}] = 0$. On the other hand, since $0 \leq nx<n \cdot \frac{1}{n} = 1$, $[nx] = 0$. Hence the equality holds.

(2) Assume that when $x_0 \in I_k$, $[x_0] + \cdots + [x_0 + \frac{n-1}{n}] = [nx_0]$. Let $x \in I_{k+1}$. Then we know that $x - \frac{1}{n} \in I_k$. By induction hypothesis we obtain

$$\begin{aligned}[x - \frac{1}{n}] + [x] + \cdots + [x - \frac{1}{n} + \frac{n-1}{n}] &= [n \cdot (n - \frac{1}{n})] \\ &= [nx - 1] \\ &= [nx] - 1\end{aligned}$$

So that

$$\begin{aligned}[x] + \cdots + [x - \frac{1}{n} + \frac{n-1}{n}] + [x + \frac{n-1}{n}] &= [nx] - 1 + [x + \frac{n-1}{n}] - [x - \frac{1}{n}] \\ &= [nx] - 1 \ \ + \ \ 1 \\ &= [nx]\end{aligned}$$

(3) By M.I, the equality holds for each natural number k, and then it holds no matter which inteval I_k x belongs to. Hence the equality holds for all $x \geq 0$.

□