Cardinal Numbers

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Let $A \approx B$ denote set equinumerosity (equipotent sets). Let $A^B = \{f \mid f : B \to A\}$. The well-definedness of the cardinal number $\alpha^{\beta+\gamma}$ follows from

Theorem 1. *If $A \approx A'$, $B \approx B'$, $C \approx C'$, $B \cap C = B' \cap C' = \varnothing$. Then $A^{B \sqcup C} \approx A'^{B' \sqcup C'}$.*

Proof. Let $f : A \to A'$, $g : B \to B'$, $h : C \to C'$ be bijections. Let Ψ map $\phi : B \sqcup C \to A$ to $\psi : B' \sqcup C' \to A'$, by $\psi(g(b)) = f(\phi(b))$ and $\psi(h(c)) = f(\phi(c))$. Ψ is one to one because once $\Psi(\phi_1) = \Psi(\phi_2)$, we show $\phi_1 = \phi_2$ by showing that $f \circ \phi_j$ are equal. Let $\Psi_j = \Psi(\phi_j)$ then everything is trivial. We show Ψ is one-to-one, and similarly there is a reverse one-to-one map and then use Schröder-Bernstein Theorem. $\square$

In fact, above well-definedness can be divided into that of $\alpha + \beta$ and that of α^β.

Theorem 2. $A^{B \sqcup C} \approx A^B \times A^C$

Proof. Let $G : B \sqcup C \to A$. We construct $g_1 : B \to A$ and $g_2 : C \to A$ by $g_1(b) = G(b)$ and $g_2(c) = G(c)$. Let $\mathfrak{G}$ be the map that maps G to (g_1, g_2). If $(g_1, g_2) := \mathfrak{G}(G) = \mathfrak{G}(H) =: (h_1, h_2)$, we check $G = H$ by: For $b \in B$, $G(b) = g_1(b) = h_1(b) = H(b)$, and for $c \in C$, $G(c) = g_2(c) = h_2(c) = H(c)$. Let $(g_1, g_2) \in A^B \times A^C$. Then it is obtainable because this $G : B \sqcup C \to A$ does: $G(b) := g_1(b)$ and $G(c) := g_2(c)$. $\square$

Theorem 3. $(A^B)^C \approx A^{B \times C}$

The two definition are the Exponential Law.

1 Latex Time !!

$\boxed{\text{Very gooooooood !!}}$

center

$$\underset{=\frac{5}{6}}{\tfrac{1}{2} + \tfrac{1}{3}} = \underbrace{1 - \frac{1}{6}}_{} = \underbrace{2 \cdot \frac{5}{12}}_{Check!}.$$

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