Equivalence Relations, Partitions, and Rationals

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(Recall that) An *equivalence relation* "$\sim$" on A is a relation satisfying the following conditions:

(i) $x \sim x$ for all $x \in A$. [Reflexivity]

(ii) If $x \sim y$ then $y \sim x$ for all $x, y \in A$. [Symmetry]

(iii) If $x \sim y$ and $y \sim z$ then $x \sim z$ for all $x, y, z \in A$. [Transitivity]

Example 1. *In $\mathbb{Z}$, we define $p \sim q$ if and only if $5|p-q$. Then $\sim$ is an equivalence relation on $\mathbb{Z}$. (In fact, $p \sim q$ means $p \equiv q$ (mod 5).)*

Proof. It clear that $5|0 = p - p$, so $p \sim p$ for all p. If $5|p-q$ then $5|(p-q)(-q) = q-p$, so $p \sim q$ implies $q \sim p$. If $5|p-q$ and $5|q-r$ then $5|(p-q)+(q-r) = p-r$, so $p \sim q$ and $q \sim r$ implies $p \sim r$. Hence $\sim$ is an equivalence relation on $\mathbb{Z}$. □

Example 2. *Let L denote all triangles in Euclidean plane. Then similarity of graphs decides an equivalence on L.*

Proof. By proportion on lengths for corresponding edges, it's clear. □

An important purpose for defining equivalence relations is their *equivalence classes*. With the properties of them, we might deal with complicated objects with a simple way.

Let $A, \sim$ as above, and $x \in A$. Define the equivalence class of x by

$$[x]_\sim = \{y \in A : \ y \sim x\}$$

and the quotient set is defined by

$$A/\sim = \{[x]_\sim \ : \ x \in A\}$$

Then we have a property:

Proposition 1. *Let $A, \sim$ as above. Then*

(a) For $x, y \in A$, if $x \sim y$ then $[x]_\sim = [y]_\sim$; if $x \nsim y$ then $[x]_\sim \cap [y]_\sim = \varnothing$.

(b) $\cup_{x \in A}[x]_\sim = A$.

In fact, this leads us to what is called a partition on a set.

Let A be a set, and Π be a collection of subsets of A. Π is called a partition on A if

(i) Whenever $P_1 \neq P_2, \in \Pi$, it follows that $P_1 \cap P_2 = \varnothing$.

(ii) $\bigcup \Pi \ (= \cup_{P\in\Pi} P) = A$.

Example 3. *If* $A := \{1,2,3,4,5,6,7,8,9\}$, *then* $\Pi := \{\,\{1,4,7\},\, \{2,5,8\}\, \{3,6,9\}\,\}$ *is a partition on* A.

Example 4. $A/\sim$ *is a partition on* A *(Why?). Sometimes, it is called the partition from* $\sim$, *and denoted* $\Pi_\sim$.

Moreover, we wonder whether we can produce a equivalence relation from a given partition? We define

$$\sim_\Pi = \{(x,y):\ x,y \in P,\ for\, some\, P \in \Pi\}$$

Then we have to properties to show:

Proposition 2. $\sim_\Pi$ *is an equivalence relation on* A. $\Pi_\sim$ *is a partition on* A.

Proof. Verification for the three condition is quite direct. We only show the later assertion. For $p \in \Pi_\sim$. Let $P = [x]_\sim$ then $P \subset A$. If $P_1 \neq P_2$, choose $P_1 = [x_1]_\sim$ and $P_2 = [x_2]_\sim$. Let (Why can we let?) $x_0 \in [x_1]_\sim \cap [x_2]_\sim$. Then $x_0 \sim x_1$ and x_2. Hence $x_1 \sim x_2$. Hence

$$P_1 = [x_1]_\sim \overset{Why?}{=} [x_2]_\sim = P_2,$$

a contradiction. Hence $P_1 \cap P_2 = \varnothing$. Let $a \in A$. Choose $[a]_\sim \in \Pi_\sim$, then $a \in [a]_\sim$. Hence $\bigcup \Pi = A$. □

Theorem 1. *Let* A, $\sim$, *and* Π *be as before. Then*

(1) $\sim_{\Pi_\sim} = \sim$.

(2) $\Pi_{\sim_\Pi} = \Pi$.

Proof. For (1), if $(a,b) \in \sim_{\Pi_\sim}$, then $a,b \in P$ for some $P \in \Pi$. By definition of $\Pi_\sim$, $P = [x]_\sim$ for some $x \in A$. So we find that $a \sim x$ and $b \sim x$. Hence $a \sim b$ and then $(a,b) \in \sim$.

Conversely, given $(a,b) \in \sim$, then $a \sim b$ and $a,b \in [a]_\sim$. Since $[a]_\sim \in \Pi$, we obtain that $a,b \in \sim_{\Pi_\sim}$. Hence we have the equality.

For (2), Let $P \in \Pi_{\sim_\Pi}$. Then $P = [x]_{\sim_\Pi}$ for some $x \in A$. Then $x \in P$. Since $x \in [x]_{\sim_\Pi}$, $x \in P'$ for some $P' \in \Pi$. Then P and P' has a common element x, so $P = P' \in \Pi$.

Conversely, for $P \in \Pi$, we can find $x \in P$. Because $x \sim_\Pi x$, $[x]_{\sim_\Pi} \in \Pi_{\sim_\Pi}$. We are going to show that

$$P = [x]_{\sim_\Pi}$$

Let $y \in [x]_{\sim_\Pi}$. Then $y \sim_\Pi x$. It follows that $x,y \in P'$ for some $P' \in \Pi$. Since $x \in P$, $P' = P$. This yields that $y \in P$. Let $y \in P$. By $x \in P$ and $P \in \Pi$ we obtain $x \sim_\Pi y$. So $y \in [x]_{\sim_\Pi}$. Hence $P = [x]_{\sim_\Pi}$, and then $P \in \Pi_{\sim_\Pi}$. □

A little proposition afterward is:

Proposition 3. *As all assumptions above, one has that*

(a) If $\Pi_{\sim_1} = \Pi_{\sim_2}$, *then* $\sim_1 = \sim_2$.

(b) If $\sim_{\Pi_1} = \sim_{\Pi_2}$, *then* $\Pi_1 = \Pi_2$.

Proof. We simply have $\sim_1 = \sim_{\Pi_{\sim_1}} = \sim_{\Pi_{\sim_2}} = \sim_2$, and $\Pi_1 = \Pi_{\sim_{\Pi_1}} = \Pi_{\sim_{\Pi_2}} = \Pi_2$. $\square$

Finally I shall mention some important senses for application of equivalence relations for pure mathematics. One example is the *rationalization* for $\mathbb{N}$ (or for an *Integral Domain*). In fact, in many other fields of mathematics such as the notion *homotopy* in *algebraic topology*, $textitp-adic\ numbers$ in *algebraic number theory*, and so on.

Example 5. *On* $\mathbb{N} \times \mathbb{N}$, *define*

$$\langle a, b\rangle \sim \langle c, d\rangle \textit{ if and only if } ad = bc$$

Then we must have that $\sim$ *is an equivalence relation on* $\mathbb{N}\times\mathbb{N}$. *Further, the quotient set* $\mathbb{N}\times\mathbb{N} \ / \ \sim$ *will become the set of rationals (In intuitive sense, it means all positive rational numbers).*

Proof. Reflexivity and symmetry are clear. We only show transitivity. If $\langle a, b\rangle \sim \langle c, d\rangle$ and $\langle c, d\rangle \sim \langle e, f\rangle$, then we have $ad = bc$ and $cf = de$. It follows that

$$daf = adf = bcf = bde = dbe$$

Hence $af = be$, and then $\sim$ is an equivalence relation. $\square$

Example 6. *Define* $\mathfrak{Q} = \mathbb{N} \times \mathbb{N} \ / \ \sim$. *Then we hope to construct its operations. Intuitively, if* $X, Y \in \mathfrak{Q}$, *expressed as* $[\langle x_1, x_2\rangle]$ *and* $[\langle y_1, y_2\rangle]$ *respectively (We omit the subscription* $\sim$ *since it contains no vagueness.), we may define*

$$X + Y = [\langle x_1y_2 + x_2y_1, x_2y_2\rangle]$$
$$Y \cdot Y = [\langle x_1y_1, x_2y_2\rangle]$$

We then ought to show that this definition is well-defined. i.e. When X *expressed as both* $[\langle x_1, x_2\rangle]$ *and* $[\langle r_1, r_2\rangle]$, *and* Y *both* $[\langle y_1, y_2\rangle]$ *and* $[\langle s_1, s_2\rangle]$, *then we must have* $[\langle x_1y_2 + x_2y_1, x_2y_2\rangle] = [\langle r_1s_2 + r_2s_1, r_2s_2\rangle]$ *and* $[\langle x_1y_1, x_2y_2\rangle] = [\langle r_1s_1, r_2s_2\rangle]$. *i.e.*

$$\langle x_1y_2 + x_2y_1, x_2y_2\rangle \sim \langle r_1s_2 + r_2s_1, r_2s_2\rangle$$
$$\langle x_1y_1, x_2y_2\rangle \sim \langle r_1s_1, r_2s_2\rangle$$

Proof. Assume $\langle x_1, x_2\rangle \sim \langle r_1, r_2\rangle$ and $\langle y_1, y_2\rangle \sim \langle s_1, s_2\rangle$. Then

$$\begin{aligned}
&(x_1y_2 + y_1x_2) \cdot r_2s_2 \\
&\quad = x_1r_2y_2s_2 + y_1s_2x_2r_2 \\
&\quad = x_2r_1y_2s_2 + y_2s_1x_2r_2 \\
&\quad = (r_1s_2 + r_2s_1) \cdot x_2y_2
\end{aligned}$$

and

$$\begin{aligned}
&(x_1y_1) \cdot (r_2s_2) \\
&\quad = x_1r_2y_1s_2 = x_2r_1s_1y_2 \\
&\quad = (r_1s_1) \cdot (x_2y_2)
\end{aligned}$$

Hence the two equivalence holds. $\square$

Note that for $x \in \mathbb{N}$, we define $\bar{x} = [\langle x, 1\rangle]$, and for $x, y \in \mathbb{N}$, we define $x/y = [\langle x, y\rangle]$.

Theorem 2. *For given $X, Y \in \mathfrak{Q}$ (expressed as q/p, n/m respectively), there is a unique $U \in \mathfrak{Q}$ such that*

$$X \cdot U = Y$$

Proof. (Uniqueness) Assume $q/p \cdot s/r = n/m$ and $q/p \cdot s'/r' = n/m$. Then we have $\langle qs, pr\rangle \sim \langle qs', pr'\rangle$. It follows that $pqr's = prqs'$. Hence $r's = rs'$, and then $s/r = s'/r'$.

(Existence) Choose $U = ps/qr$. Then $X \cdot U = q/p \cdot ps/qr = qps/pqr$. Because $qps \cdot r = pqr \cdot s$, we obtain $X \cdot U = s/r$. □

As a consequence, we define $\frac{Y}{X}$ as the unique U in the previous theorem.

Recall that at first we begin with the pair $\langle x, y\rangle$, and by eauivalence classes we define x/y. Moreover, by above equation, we introduce the notation $\frac{N}{M}$. Now those x's in $\mathbb{N}$ and those $\bar{x}$ looks alike, so we hope to illustrate the relationship between these two types of numbers.

The sense is that, we are used to denoting it ($\{\bar{x} : x \in \mathbb{N}\}$) by $\mathfrak{N}$, and view it the *exact* natural number set we always write and use. Hence

$$\mathfrak{N} \subset \mathfrak{Q}$$

while the set $\mathbb{N}$ is from now on hidden behind. Hence we have the following property.

Example 7. *For each $X \in \mathfrak{Q}$, there are $M, N \in \mathfrak{N}$ such that*

$$X = \frac{N}{M}$$

Proof. It's clear that we can write X in the following form

$$X = [\langle n, m\rangle] = n/m$$

By easy verification we obtain that

$$M \cdot n/m = \bar{m} \cdot n/m = \bar{n} = N$$

Hence $n/m = \frac{N}{M}$. □

This is merely a simple introduction about construction of those rationals. There are many properties that I have not mentioned, for example, ordering, but all discussion above reveals enough information for those basic properties for rationals. If someone feels interested on this topic, a reference is *Foundations of Analysis* written by *Edmund Landau*.