Cardinality

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To consider the size of a set, the most important thing is about functions. If $f : A \to B$ is a function, the following properties are elementary:

1 (One-to-one)**.** *Supposely that whenever $f(x) = f(y)$, it follows that $x = y$, then the function f is called one-to-one from A to B.*

2 (Onto)**.** *Supposely that for each $b \in B$, there is an $a \in A$ such that $f(a) = b$, then this function f is called from A onto B.*

3. *An Injection (injective function) is an one-to-one function; a surjection (surjective function) is an onto function; a bijection (bijective function) is an injective and surjective function.*

4 (Equinumerosity)**.** *The sets A, B are called equinumerous (of the same cardinality), denoted by $A \approx B$, if there is a bijection from A to B; $A \preceq B$ (of (weakly) less cardinality than) if there is an injection from A to B. $A \prec B$ (of strickly less cardinality than) if $A \preceq B$ but $A \not\approx B$. Sometimes, equinumerous is replaced by equipotent.*

1 Bijections

Now I think it is necessary to verify that if those elementary functions are one-to-one and onto between each pair of given sets.

5. *$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \pi x$ is one-to-one and onto because,*

(i) For $x, y \in \mathbb{R}$, if $f(x) = f(y)$, i.e. $\pi x = \pi y$, then $x = y$ immediately.

(ii) For $y_0 \in \mathbb{R}$, choose $x = \frac{y_0}{\pi}$. Then $f(x) = \pi \cdot \frac{y_0}{\pi} = y_0$.

6. *$g : \mathbb{R}^+ \to \mathbb{R}^+,\ x \to x^2$ is a bijective.*

(i) For $a, b \in \mathbb{R}^+$, if $a^2 = b^2$ then $(a-b)(a+b) = 0$. Because $a + b \neq 0$, we obtain $a - b = 0$, i.e. $a = b$.

(ii) For $b_0 \in \mathbb{R}^+$, we choose $a = \sqrt{b_0}$. Then $g(a) = (\sqrt{b_0})^2 = b_0$.

Hence, g is bijective.

7. *The function $u(x) = x^3$ in $\mathbb{R}$ is one-to-one and onto.*

Proof. For $y \in \mathbb{R}$, choose $x = \sqrt[3]{y}$. This implies that u is onto. If $x, y \in \mathbb{R}$ such that $x^3 = y^3$. Then consider

$$\begin{aligned} 0 = x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\ &= (x-y)\left((x + \frac{1}{2}y)^2 + \frac{3}{4}y^2\right) \end{aligned}$$

If $(x-y)\left((x+\frac{1}{2}y)^2 + \frac{3}{4}y^2\right) = 0$, then $x = y$. $(= 0)$. If $(x-y)\left((x+\frac{1}{2}y)^2 + \frac{3}{4}y^2\right) \neq 0$, then $x - y = 0$. Hence u is one-to-one. □

Another way to show "one-to-one" is that we have several cases according as $x>0$, $x = 0$, or $x<0$ and, $y>0$, $y = 0$, or $y<0$. For example, if x, y both >0 (<0) then $x^2 + xy + y^2 > 0$. Hence $x^3 - y^3 = 0$ implies $x = y$.

This division inspires us a helpful property.

8. *If $A \cap B = C \cap D = \varnothing$, and $f : A \cup B \to C \cup D$ is*

(i) one-to-one from A onto C and

(ii) one-to-one from B onto D,

then f is a bijection.

This is a trivial statement, so we omit the proof.

Still another example is a trigonometric function.

9. *Show that $G(x) = \sin x$, where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is one-to-one from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(-1, 1)$.*

Proof.

(one-to-one) If $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $G(x) = G(y)$. Then

$$0 = \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

Since $\frac{x+y}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos\left(\frac{x+y}{2}\right) \neq 0$, it follows that

$$\sin\left(\frac{x-y}{2}\right) = 0.$$

Hence $x = y$.

(onto) Let $K \in (-1, 1)$. Since

(i) $G(x) = \sin x$ is continuous in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\sin(-\frac{\pi}{2}) = -1$, $\sin(\frac{\pi}{2}) = 1$.

(iii) $-1<K<1$.

Intermediate Value Theorem hence tell us that there is a $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$G(c) = \sin c = K.$$

$\square$

Exercise 1. *Check* $\tan x$ *is a bijection from* $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

2 Some Properties and Number Systems

Next, our purpose is to show that $\mathbb{N} \approx \mathbb{Z} \approx \mathbb{Q} \prec \mathbb{R} \approx \mathbb{C}$.

2.1 $\mathbb{N}$ and $\mathbb{Z}$

10. $\mathbb{N} \approx \mathbb{Z}$

Proof. We ought to find a bijection. Let

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ -\frac{n-1}{2}, & \text{otherwise.} \end{cases} \tag{1}$$

Then it is routine to show bijectivity.

For even natural numbers x, y, if $f(x) = f(y)$, i.e. $\frac{x}{2} = \frac{y}{2}$, then $x = y$. Next, for $p \in \mathbb{Z}$, $p>0$, we find that $f(2p) = \frac{2p}{2} = p$; For odd natural numbers x, y, if $f(x) = f(y)$, i.e. $-\frac{x-1}{2} = -\frac{y-1}{2}$, then $x = y$. Similarly, if $p \in \mathbb{Z}$, $p \leq 0$, then $f(-2p+1) = -\frac{(-2p+1)-1}{2} = p$. By previous example, f is bijective. $\square$

2.2 $\mathbb{N}$ and Fractions

For the goal of the fact that $\mathbb{Z} \approx \mathbb{Q}$, we need quite a few effort. Firstly, we embed a significant property into our discussion.

11. *Let* Λ *be an index set. Given* $f : \sqcup_{j\in\Lambda} A_j \to \sqcup_{j\in\Lambda} C_j$*, if for any* $j \in \Lambda$*,* f *is one-to-one from* A_j *onto* C_j*, then* f *is one-to-one and onto.*

Note that the notation $\sqcup_{j\in\Lambda} S_j$ *means disjoint union. If the sets* S_j*'s are pairwise disjoint, we write* $\sqcup_{j\in\Lambda} S_j$ *for their union instead.*

The statement is useful in the following property, by which we will show that $\mathbb{N} \approx \mathbb{Q}^+$.

12. $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$.

Proof. Define

$$\widehat{n} = \begin{cases} 0, & \text{if } n \leq 0,\ n \in \mathbb{Z}; \\ 1 + 2 + 3 + \cdots + n, & \text{if } n \in \mathbb{N}. \end{cases} \tag{2}$$

Let $p(m,n) = \widehat{m+n-2} + m$. We're going to show that p gives a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. If $m + n = k \in \mathbb{N}$, we hope to verify that f is bijective from

$$A_k := \{\langle m, n\rangle \ : \ m + n = k\}$$

to

$$C_k := (\widehat{k-2}, \widehat{k-1}] \cap \mathbb{N}.$$

Let k be given.

(1) For $\langle m, n\rangle \in A_k = \{\langle 1, k-1\rangle, \langle 2, k-2\rangle, \cdots, \langle k-1, 1\rangle\}$,

$$\begin{aligned} \widehat{k-2} = \widehat{m+n-2} < p(m,n) &= \widehat{m+n-2} + m \\ &= \widehat{k-2} + m \le \widehat{k-2} + (k-1) = \widehat{k-1} \end{aligned}$$

This means p maps A_k into C_k.

(2) Let $\langle m_1, n_1\rangle$, $\langle m_2, n_2\rangle \in A_k$. Suppose that $p(\langle m_1, n_1\rangle) = p(\langle m_2, n_2\rangle)$, then

$$\widehat{k-2} + m_1 = \widehat{k-2} + m_2$$

So $m_1 = m_2$, and $n_1 = n_2$. This indicates, that f is one-to-one from A_k to C_k.

(3) Let $N \in C_k$. Denote

$$N = \widehat{k-2} + j$$

Choose $\langle m, n\rangle = \langle j, k-j\rangle$. Then $p(m,n) = N$.

Since $\mathbb{N} \times \mathbb{N} = \sqcup_{k=2}^{\infty} A_k$, $\mathbb{N} = \sqcup_{k=2}^{\infty} \left((\widehat{k-2}, \widehat{k-2}] \cap \mathbb{N} \right)$, which satisfies all conditions of previous example. Hence p is bijective. □

Our next mission is that $\mathbb{Q}^+ \preceq \mathbb{N} \times \mathbb{N}$. Since we already have $\mathbb{N} \preceq \mathbb{Q}^+$, we'll show that $\mathbb{N} \approx \mathbb{Q}^+$.

13. $\mathbb{Q}^+ \preceq \mathbb{N} \times \mathbb{N}$ *because, we may choose* $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ *by*

$$f(\frac{q}{p}) = \langle q, p\rangle \qquad with\, gcd(p,q) = 1,$$

i.e.

$$f = \{\langle \ \frac{a}{b}, \langle a, b\rangle \ \rangle : a, b \in \mathbb{N}, \ gcd(a,b) = 1\}$$

i.e.

$$\frac{n}{m} \mapsto \langle \frac{n}{gcd(n,m)}, \frac{m}{gcd(n,m)}\rangle \qquad for\, m, n \in \mathbb{N}.$$

To show that f is injective, given $\frac{m_1}{n_1}, \frac{m_2}{n_2} \in \mathbb{Q}^+$ such that $gcd(m_1, n_1) = gcd(m_2, n_2) = 1$ and $f(\frac{m_1}{n_1}) = f(\frac{m_2}{n_2})$. Since $\langle m_1, n_1\rangle = \langle m_2, n_2\rangle$, it follows that $m_1 = m_2$ and $n_1 = n_2$.

2.3 Fractions to $\mathbb{Q}^+$ and $\mathbb{Q}$

14. *If $B \approx \mathbb{N}$, $A \subset B$, and $\mathbb{N} \preceq A$, then $A \approx \mathbb{N}$.*

Proof. Denote

$$B = \{b_1, b_2, \cdots\}$$

Then we hope to construct a sequence representing A, i.e. $A = \{y_1, y_2, \cdots\}$.

Let n_1 be the smallest integer n such that $b_n \in A$, and write $A_1 = A \setminus \{n_1\}$.

Let n_2 be the smallest integer $n{>}n_1$ such that $b_n \in A_1$, and write $A_2 = A_1 \setminus \{n_2\}$.

Let n_3 be the smallest integer $n{>}n_2$ such that $b_n \in A_2$, and write $A_3 = A_2 \setminus \{n_3\}$.

$\vdots$

The process can repeat permanently because $\mathbb{N} \preceq A_j$ for all $j \in \mathbb{N}$ (Why?). Since $A_j \neq \varnothing$, n_{j+1} can be chosen (from A_j). Next, to show

$$A = B_0 := \{b_{n_1}, b_{n_2}, \cdots\}$$

$B_0 \subset A$ is trivial. Let $a \in A \subset B$. Then $a = b_m$ for some $m \in \mathbb{N}$. Because $1 \leq n_1{<}n_2{<}\cdots$, we know $m \in (n_j, n_{j+1}]$. It follows that $a = b_{n_{n+1}}$. Hence $A \subset B_0$. Therefore $A = B_0$, and then $A \approx \mathbb{N}$. □

The following two propositions are prepared for the third proposition.

15. *If $A \preceq C$, then $A \approx B$ for some $B \subset C$. The reason is that, if f is one-to-one from A to C, then f is from A onto $B := f(A) \subset C$.*

16. *If $A \preceq B_1$ and $B_1 \approx B_2$, then $A \preceq B_2$. We derive this as follow: If f is one-to-one from A to B_1 and g is a bijection from B_1 to B_2, then $g \circ f$ is injective from A to B_2.*

17. $\mathbb{N} \approx \mathbb{Q}^+$.

Proof. We know that $\mathbb{Q}^+ \preceq \mathbb{N} \times \mathbb{N}$. Then $\mathbb{Q}^+ \approx N$ for some $N \subset \mathbb{N} \times \mathbb{N}$. The result is immediate from preceding examples. □

18. *We can now show equinumerosity between $\mathbb{Z}$ and $\mathbb{Q}$. Since we already have a one-to-one function f from $\mathbb{N}$ onto $\mathbb{Q}^+$. Let $F : \mathbb{Z} \to \mathbb{Q}$, by*

$$F(x) = \begin{cases} f(x), & \text{if } n \in \mathbb{N}; \\ 0, & \text{if } x = 0; \\ -f(-x), & \text{if } x \in -\mathbb{N}. \end{cases} \tag{3}$$

Then F is bijective.

2.4 $\mathbb{Q}$ and $\mathbb{R}$

We're going to show a more astonishing property, that $\mathbb{Q}$ contains (strictly) less elements than $\mathbb{R}$. Before reaching a precise proof, we need some preparation.

19. *If $A \approx B$, $B \approx C$, then $A \approx C$. A simple proof is: Suppose that f, g are bijections between A, B and C. Then $g \circ f$ is bijective from A to C.*

20. $\mathbb{R}^+ \approx \mathbb{R}$ *because, the functions $f(x) := e^x$ and*

$$g(x) = \begin{cases} x+1, & \text{if } x \geq 0; \\ \frac{-x}{1-x}, & \text{if } x<0. \end{cases} \tag{4}$$

are one-to-one from $\mathbb{R}$ to $\mathbb{R}^+$.

21. $\mathbb{N} \prec \mathbb{R}^+$.

Proof. It's clear that $\mathbb{N} \preceq \mathbb{R}^+$. Assume that $\mathbb{R}^+ = \{f_0, f_1, f_2, \cdots\}$, written in a sequence. We list the elements below in decimal expression.

$$\begin{aligned} f_0 &= f_{00}.f_{01}f_{02}f_{03}f_{04}\cdots \\ f_1 &= f_{10}.f_{11}f_{12}f_{13}f_{14}\cdots \\ f_2 &= f_{20}.f_{21}f_{22}f_{23}f_{24}\cdots \\ f_3 &= f_{30}.f_{31}f_{32}f_{33}f_{34}\cdots \\ &\vdots \end{aligned}$$

The main idea is to construct a decimal number which is none of the f_j's. Denote $f_\infty = f_{\infty 0}.f_{\infty 1}f_{\infty 2}f_{\infty 3}\cdots$, where

$$f_{\infty j} = \begin{cases} 7, & \text{if } f_{jj} = 3; \\ 3, & \text{Otherwise.} \end{cases} \tag{5}$$

Then $f_\infty \in \mathbb{R}^+$. It is some f_k, but $f_{\infty k} \neq f_{kk}$, a contradiction. Hence $\mathbb{N} \not\approx \mathbb{R}^+$. This means $\mathbb{N} \prec \mathbb{R}^+$. □

22. *We can now show that $\mathbb{Q} \prec \mathbb{R}$. It's well-known that $\mathbb{Q} \preceq \mathbb{R}$. Assume that $\mathbb{Q} \approx \mathbb{R}$. Since $\mathbb{N} \approx \mathbb{Q}$ and $\mathbb{R} \approx \mathbb{R}^+$, it follows that $\mathbb{N} \approx \mathbb{R}^+$, a contradiction.*

2.5 $\mathbb{R}$ to $\mathbb{C}$

The last thing we hope to verify is that $\mathbb{R} \approx \mathbb{C}$. Similarly as before, for convenience, we divid it into several parts.

23. $(0, 1) \approx (0, 1) \times (0, 1)$.

To give a precise proof, we need the help of **Schröder-Bernstein Theorem**, i.e. if $A \preceq B$ and $B \preceq A$ then $A \approx B$.

Proof. The map $x \mapsto (x, \frac{1}{2})$ is an injection. Define $f : (0,1) \times (0,1) \to (0,1)$ by

$$(a, b) := (0.a_1a_2a_3 \cdots\ ,\ 0.b_1b_2b_3 \cdots) \mapsto 0.a_1b_1a_2b_2a_3b_3 \cdots$$

where none of a, b has consecutive 9's, i.e. neither a nor b is in the form $0.d_1d_2 \cdots d_j9999 \cdots$. Then if $a = 0.a_1a_2 \cdots$, $b = 0.b_1b_2 \cdots$, $c = 0.c_1c_2 \cdots$ and $d = 0.d_1d_2 \cdots$ are such that $f(a, b) = f(c, d)$, we write

$$0.a_1b_1a_2b_2 \cdots = 0.c_1d_1c_2d_2 \cdots$$

According to uniqueness of decimal expression, it follows that $a_j = c_j$ and $b_j = d_j$. Hence f is an injection. The property is then proved by Schr$\ddot{o}$der-Bernstein Theorem. □

Note that the function f is not onto. In fact, we can find no pair mapping to the number $0.010191919191 \cdots$ in $(0, 1)$, hence we need to show *double injectivity.*

The following is a slight modification to our final goal.

24. *For* $(a, b) \subset \mathbb{R}$, $(a, b) \approx (0, 1)$. *The map can be chosen as* $f(x) = (b - a)x + a$.

25. $(0, 1) \approx \mathbb{R}$. *A bijection between them can be chosen as*

$$f(x) = \frac{x}{1 + |x|}$$

from $\mathbb{R}$ *onto* $(-1, 1)$ *or*

$$g(x) = \tan x$$

from $(-\frac{\pi}{2}, \frac{\pi}{2})$ *to* $\mathbb{R}$.

26. $(0, 1) \times (0, 1) \approx \mathbb{R} \times \mathbb{R}$.

Proof. Let $f : (0, 1) \to \mathbb{R}$ be bijective. Then

$$F(a, b) = (f(a), f(b)) \quad on\, (0, 1) \times (0, 1)$$

is a bijection that we desire. □

27. *Since* $\mathbb{R} \approx (0, 1)$, $(0, 1) \approx (0, 1) \times (0, 1)$, $(0, 1) \times (0, 1) \approx \mathbb{R} \times \mathbb{R}$, *we conclude that* $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

28. $\mathbb{R} \times \mathbb{R} \approx \mathbb{C}$, *because we have the function* $k(x, y) = x + iy$. *Therefore we obtain the final equinumerosity*

$$\mathbb{R} \approx \mathbb{C}.$$

3 Countable and Uncountable Infinities

Although we've shown that the five basic sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ have only 2 levels about their amounts of elements, yet we even have no idea about what an infinite set really is. Our intuition might tell us that a finite set must contain some n elements. That means, it must be equinumerous to some *standard set* which we consider to have n elements. In the following discussion, this standard set will naturally be chosen as $I_n := \{1, 2, 3, \cdots, n\}$.

Except for the notion of an infinite set, we have another notion called countable sets. The difference from countability depends on whether the given set can be listed in a (finite of infinite) sequence.

29. *We now make a convention that* $\mathbb{W}$ *means the whole number set. i.e.* $\mathbb{W} = \mathbb{N} \cup \{0\}$. *Define* $I_n := \{1, 2, \cdots, n\} = \{k \in \mathbb{N} : 1 \leq k \leq n\}$, *and* $I_0 = \varnothing$, *for convenience of later consideration.*

30. *A set* S *is called finite if* $S \approx I_n$ *for some* $n \in \mathbb{W}$. *If* S *is not finite, then* S *is called infinite. Note that the "new" notion leads to no contradiction about what are discussed before because we did not mention the word "infinite" previously.*

31. *A set* K *is called countable if all elements of* K *are in a (finite or infinite) sequence, i.e. if* $K \approx I_n$ *for some* $n \in \mathbb{W}$ *or* $K \approx \mathbb{N}$. *In the later case,* K *is also called denumerable. So* K *is countable if* K *is either finite or denumerable. If* K *is not countable, we say* K *is uncountable.*

There are many "trivial" statements concerning infinite sets, but not all of them is easy to prove.

32. *If* S *is an infinite set, and* $T \subset S$, *then either* T *or* $S \setminus T$ *is infinite.*

Proof. Assume that S and $S \setminus T$ are both finite. Let $f : I_n \to S$ and $g : I_m \to S \setminus T$ be bijections. Then define $\varphi : I_{m+n} \to S$ as follow:

$$\varphi(x) = \begin{cases} f(x), & \text{if } x = 1, 2, 3 \cdots, n; \\ g(x - n), & \text{if } x = n+1, n+2, \cdots, n+m. \end{cases} \tag{6}$$

Then φ is one-to-one from I_{n+m} onto S, a contradiction to the fact that S is infinite. ☐

33. *The union of a finite set and a denumerable set is denumerable because, if the finite set is expressed as* $\{a_1, a_2, \cdots, a_n\}$ *while the other* $\{b_1, b_2, \cdots\}$, *then the union* $\{a_1, \cdots, a_n, b_1, b_2, \cdots\}$ *is equinumerous to a subset (why not exactly ?) of* $\{\langle a, 1\rangle, \cdots \langle a, n\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \cdots\}$. *Moreover,* $\mathbb{N} \preceq$ *the union, we conclude that* $\mathbb{N}$ *is equinumerous to the union.*

34. *The disjoint union of two denumerable sets are denumerable. Therefore, the disjoint union of* n *denumerable sets are also denumerable.*

Proof. It easy to check that $\mathbb{N}$ is equinumerous to $2\mathbb{N}$ and $2\mathbb{N}+1$ (We define $2\mathbb{N} = \{2, 4, 6, 8, \cdots\}$ and $2\mathbb{N}+1 = \{1, 3, 5, \cdots\}$.). Let A, B be two disjoint sets. Choose $f : 2\mathbb{N} \to A$, $g : 2\mathbb{N}+1 \to B$ as bijections. Define $G : \mathbb{N} \to A \sqcup B$ by

$$G(x) = \begin{cases} f(x), & \text{if } x \text{ is even}; \\ g(x), & \text{if } x \text{ is odd}. \end{cases} \tag{7}$$

Then G shows that $A \sqcup B$ is denumerable. ☐

35. *How about the union of n denumerable sets?*
How about a denumerable union of denumerable sets?

For the second assertion, we need some observation.

36. *The denumerable union of pairwise disjoint denumerable sets are denumerable.*

Proof. Denote these sets by $S_1, S_2, \cdots$, where

$$\begin{aligned} S_1 &= \{s_{11}, s_{12}, s_{13}, \cdots\} \\ S_2 &= \{s_{21}, s_{22}, s_{23}, \cdots\} \\ S_3 &= \{s_{31}, s_{32}, s_{33}, \cdots\} \\ &\vdots \end{aligned}$$

are their elements. Then the mapping $\langle i, j\rangle \leftrightarrow s_{ij}$ is bijective. Hence $\mathbb{N} \approx \bigcup_{n\in\mathbb{N}} S_n$. □

37. *A set S is countable if and only if $S \preceq \mathbb{N}$.*

Proof. If S is denumerable, then $S \approx \mathbb{N}$ and hence $S \preceq \mathbb{N}$. If S if finite, then $S \preceq \mathbb{N}$ because $I_n \preceq \mathbb{N}$. Conversly, Let f be one-to-one from S into $\mathbb{N}$. Let $T = Im(f)$. Then

(a) If T contains no maximum, then $\mathbb{N} \preceq T$ (why?). Since $T \subset \mathbb{N}$, we obtain that $\mathbb{N} \approx T$. Since f is onto T, it follows that $S \approx T$.

(b) If T contains a maximum, say $n_0 \in \mathbb{N}$. Then $T \subset I_{n_0}$. Hence T is equinumerous to some I_m (why? This might be proved by induction.). Similarly, the assertion is proved by $S \approx T \approx I_m$.

□

38. *The denumerable union of pairwise disjoint countable sets are countable.*