On Axiom of Replacement

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In this note I shall write down what it is then after the Axiom of Replacement (AoR).

The introduction of this axiom enables the following Expressional Function Theorem (EFT). In fact, with a little adjustment, EFT is equivalent to AoR. A more important theorem is the transfinite induction theorem (TIT).

- (AoP). Let A be a set. Supposely for sets $x \in A, y$, and z, $y = z$ whenever $\varphi(x, y)$ and $\varphi(x, z)$, then, there is a set B such that for a set t, $t \in B$ if and only if $\varphi(x, t)$ for some $x \in A$.
- (EFT). There is a unique function $f : A \to \cdot$ with $x \mapsto \mathcal{O}(x)$.
- (TIT). Let $(A, \leq)$ be a well ordered set. Suppose that for any function f there is a unique set y such that $\gamma(f, y)$. Then there is a unique $F : A \to \cdot$ such that $\gamma(F|_{\text{seg } t}, F(t))$.

In other words, supposing that a predicate $\Gamma(f)$ is given, there is a unique $F : A \to \cdot$ such that $F(t) = \Gamma(F|_{\text{seg } t})$.

Now we let $A = \mathbb{W}$, and $\Gamma(\varnothing) = a$, and $\Gamma(f) = \mathcal{O}(f(sup(dom(f))))$ ($= \varnothing$ while undefined). Then TIT says that there is an $F : \mathbb{W} \to \cdot$ such that $F(t) = \Gamma(F|_{\text{seg } t})$. Let us observe F.

$$\begin{aligned} F(0) &= \Gamma(F|_{\text{seg } 0}) = \Gamma(\varnothing) = a. \\ F(t) &= \Gamma(F|_{\text{seg } t}) = \Gamma(F|_{\{0,1,2,3,\ldots,t-1\}}) = \mathcal{O}(F|_{\text{seg } t}(sup(dom(F|_{\{0,1,2,3,\ldots,t-1\}})))) \\ &= \mathcal{O}(F|_{\text{seg } t}(sup(\{0,1,2,3,\ldots,t-1\}))) = \mathcal{O}(F|_{\text{seg } t}(t-1)) = \mathcal{O}(F(t-1)). \end{aligned}$$

This indicates, this is a sequence

$$\{a, \mathcal{O}a, \mathcal{O}\mathcal{O}a, \mathcal{O}\mathcal{O}\mathcal{O}a, \cdots\}.$$

Moreover, we specify the theorem to the fact of the existence of this sequence.

With the help of TIT, in fact, the following two existences are equivalent, with different ways to reach Peano's Axioms and hence the structure of $\mathbb{N}$ and $\mathbb{W}$:

(i) The set $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \cdots\}$.

(ii) The set $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \cdots\}$.

Note that the superiority of the later one from the former one occurs in the theory of ordinal numbers (ordinal types). In fact, all that is in the later set are ordinals but the former set fails.