A Quite Note On Set Theory

1 Sets

1.1 Basic Sets

Collected objects are often what mathematics focus on, so we have the notion of sets. A collection of objects "is called" a **set**. We care about the members of sets. Sets A and B "are called" **equal** if and only if they contain the same elements, by which I mean,

whenever $x \in A$, we have $x \in B$, and
whenever $y \in B$, we have $y \in A$.

1.2 Operations

Having sets, we hope to define operations on them.
DEFINITION. For sets A, B,
(i) $A \cap B := \{x \,|\, x \in A \text{ and } x \in B\}$;
(ii) $A \cup B := \{x \,|\, x \in A \text{ or } x \in B\}$;
(iii) $A \setminus B := \{x \,|\, x \in A \text{ but } x \notin B\}$.

Some properties are immediately.
PROPOSITION. Let S, T, U be sets.
(a) $(S \cap T) \cap U = S \cap (T \cap U)$.
(b) $(S \cup T) \cup U = S \cup (T \cup U)$.
(c) $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$.

PROOF. I only show (c). Let $x \in S \cup (T \cap U)$. Then $x \in S$ or $x \in T \cap U$. Our goal is to prove that $x \in S \cup T$ and $x \in S \cup U$. If $x \in S$, then $x \in S \cup T$ by definition of union. If $x \in T \cap U$, then $x \in T$ (definition of intersection). So $x \in S \cup T$ by definition of union.

Let $y \in (S \cup T) \cap (S \cup U)$. Then $y \in S \cup T$ and $y \in S \cup U$. We want to show that $y \in S$ or $y \in T \cap U$. If $y \in S$, then we finish the proof. Suppose that $y \notin S$. We want to show $y \in T$ and $y \in U$.

For $y \in T$, since $y \in S \cup T$ but $y \notin S$, by definition of union, we get that $y \in T$. Similarly, $y \in U$. Thus, by convention of set equality, we've shown the identity. $\square$

1.3 Specified Sets

Let A be a set. We call S a **subset** set of A if for any $x \in S$, it holds that $x \in A$. The **power set** of A is defined as

$$\wp(A) = \{S \,|\, S \subset A\}.$$

1.4 Ordered Pairs and Products

The ordered pair is another concept different from that of a set. We denote an **ordered pair** by (a, b). To distinguish an ordered pair from a set, note that ordered pairs have the property that

$$\text{If } (a, b) = (c, d) \quad \text{then } a = c\ ,\ b = d.$$

Similarly we also have the form $(a_1, \cdots, a_k)$ which contains k components. We omit the detail.

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2 Relations

2.1 Basic Form

A **relation** between a set A and a set B is a subset R of $\wp(A \times B)$. By the notion of relation we hope to classify or build relationship between given sets. Note that we care about those relations with some specified properties.

Let R be a relation on A (i.e. a relation between A and itself).

(R) Reflexivity: If $(x,x) \in R$ for any $x \in A$, then we say R is reflexive.

(S) Symmetry: Suppose that if $(x,y) \in R$ then $(y,x) \in R$. Then we say R is symmectric.

(A) Anty-Symmetry: Suppose that if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$. Then we say R is anty-symmectric.

(T) Transitivity: Suppose that if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. Then we say R is transitive.

Example. $R_1 := \{(x,y) \in \mathbb{N} \times \mathbb{N} \,|\, x \leq y\}$, $R_2 := \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \,|\, m \equiv n \,(\text{mod }) 5\}$, $R_3 := \{(p,q) \in \mathbb{N} \times \mathbb{N} \,|\, p \text{ divides } q\}$, and $R_4 = \{(S,T) \,|\, S \subset T,\ S,T \subset \mathbb{R}\}$ are examples of each type above.

For writing and reading convenience, we always write

$$xRy$$

instead of $(x,y) \in R$.

2.2 Equivalence Relations

An **equivalence relation** is a reflexive, symmetric, transitive relation on some A. It is usually denoted by the symbol $\sim$. To be explicit, $\sim$ is an equivalence relation means: For each $x, y, z \in A$,

(i) $x \sim x$.

(ii) if $x \sim y$, then $y \sim x$.

(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example. $\sim_1 := R_3 = \{(p,q) \in \mathbb{N} \times \mathbb{N} \,|\, p \text{ divides } q\}$ is an equivalence relation on $\mathbb{Z}$.

Example. $\sim_2 := \{(\langle x,y\rangle, \langle z,w\rangle) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,|\, xw = yz\}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. Moreover, we observe that $\langle 1,2\rangle \sim \langle 2,4\rangle \sim \langle 3,6\rangle$, and $\langle 2,5\rangle \sim \langle 4,10\rangle \sim \langle 100,250\rangle$, which presents the notion of the rationals.

2.3 Equivalence classes

However, an important further observation is the classification on those elements in A by equivalence.

We make the definitions.

Definition. On a relation R, the **domain**, and the **range**, of R are defined as: $dom(R) = \{x \in A \,|\, (x,y) \in R \text{ for some } y \in A\}$ and $ran(R) = \{y \in A \,|\, (x,y) \in R \text{ for some } x \in A\}$.

Note that we may also define the domain and the range, or a relation R between sets A and B. However, this is detail.

For an equivalence relation $\sim$ on a set A, given $x \in A$, we denote the equivalence class by $[x]_\sim$, sometimes omitting the index $\sim$, which is defined as

$$[x]_\sim = \{y \in A \,|\, y \sim x\}.$$

and the set of all equivalence classes of $\sim$ on A is called the **quotient set** of $\sim$ under A, denoted by $A/\sim$.

Example. On $\mathbb{Z}$, using $\sim_3 := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \,|\, 3|x-y\}$, which can be shown to be an equivalence relation. The equivalence classes are:

$$3\mathbb{Z} = \{\cdots, -3, 0, 3, 6, 9, \cdots\}$$
$$3\mathbb{Z}+1 = \{\cdots, -2, 1, 4, 7, 10, \cdots\}$$
$$3\mathbb{Z}+2 = \{\cdots, -4, -1, 2, 5, 8, \cdots\}.$$

Thus, $\mathbb{Z}/\sim_3 = \{3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2\}$.

2.4 Partitions

This leads to the notion of a partition P on the given set A. Let $P = \{P_i\}_{i \in \mathscr{A}}$, where $P_i \subset A$ for all i. P is called a **partition** of A if

(1) all sets in P are *pairwisely disjoint* and
(2) $A = \bigcup_{P_i \in P} P_i$.

PROPOSITION. (1) If $\sim$ is an equivalence relation on A, then $A/\sim$ is a partition on A.

(2) If P is a partition on A, then

$$\begin{aligned} \sim_P := &\{(x, y) \,|\, x, y \in P_i \text{ for some } P_i\} \\ = &\{(x, y) \,|\, x, y \text{ are in the same } P_i \\ &\text{for some } P_i\} \end{aligned}$$

is an equivalence relation.

3 Functions

3.1 Definition

A function is a relation which has a "unique correspondence". Recall that, in the sense of greatnness, 7 is related to 6,5,4,3. However, functions do not allow such a case. We mean that, functions is retricted to avoid multiple correspondences.

DEFINITION. (1)A function f is a relation xuch that ,whenever $(x, y) \in f$ and $(y, z) \in f$, it holds that $y = z$.

(2) A function f (defined on A) [from A to B] is a function such that ($dom(f) = A$) [$dom(f) = A$ and $ran(f) \subset B$].

There're some customary types for letting a function. The following are examples.

ORIGINAL TYPE: $f_1 := \{(x, 2x) \,|\, x \in \mathbb{R}^+\}$.
PREDICATE TYPE: $f_2(x) = 2x$, for $x \in \mathbb{R}^+$.
ASSIGNMENT TYPE: $f_3 : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto 2x$.

PROPOSITION. Two functions are equal if and only if they have the same domain and their assignments are equal for each element of the common domain.

3.2 One-to-one and Onto

There are some special types of functions.

DEFINITION. f is one-to-one if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$. f is onto if for $y \in B$, there is an $x \in A$ such that $f(x) = y$.

3.3 Inverse

In the sense of finding a solution of a certain function, we need what is called the inverse of this function. We now give the definition in the viewpoint of its essence of being a relation.

DEFINITION. Let f be a function. If $\grave{f}$ is also a function, then f is called invertable.

The main properties are:

PROPOSITION. (a) f is invertable if and only if f is one-to-one and onto,

(b) which holds if and only if there is another function g such that $g(f(x)) = f(g(x)) = x$ for all proper x (i.e. all x such the terms above are defined). (Require the AC)

3.4 The Index

It's time to illustrate the index-sets.

4 Cardinality

4.1 Equinumerosity

A natural thought about "numbers" comes from certain classificatiion of collections of objects. We have the intuition in mind, that the string "1572xc" has 6 characters. Experience tells us that counting is a correspondence between the observing set and a standare "base" set.

To be more precisely, according to my intuition, to say $\{\mathbb{N}, (2, 3), 1, 7, \frac{1}{\sqrt{10}}$ has 5 elements, I need to make an one-to-one correspondence between $\{1, 2, 3, 4, 5\}$ and $\{\mathbb{N}, (2, 3), 1, 7, \frac{1}{\sqrt{10}}$.

The first notion of counting is equinumerosity.

Definition. The sets A and B are called equinumerosity, if there is an one-to-one correspondence between A and B, and it's notation is $A \approx B$.

Proposition. Let $\mathscr{A}$ be a collection of sets. Write

$$\sim := \{(A, B) \in \mathscr{A} \times \mathscr{A} \mid A \approx B\}.$$

Then $\sim$ is an equivalence relation on $\mathscr{A}$.

A **question** is that: Why don't we let A to be the collection of all sets? The answer is that: such a set does not exist.

Assume that we have a set B which contains all sets. Then we construct another set

$$Q := \{x \in B \mid x \notin x\}.$$

If $Q \in Q$, then $Q \in B$ and $Q \notin Q$, a contradiction. If $Q \notin Q$, since $Q \in B$ (definition of B), it holds that $Q \in Q$, a contradiction. Now, neither $Q \in Q$ nor $Q \notin Q$ holds, which is another contradiction. Hence such a B fails to exist. □

Definition. If there is a function f from A one-to-one to B, then we say that B dominates A, or A is dominated by B, written $A \precsim B$.

4.2 The Integers

Example. (a) $\mathbb{N} \approx \mathcal{O}_{\mathbb{N}} := \{2n - 1 \mid n \in \mathbb{N}\}$. (b) $\mathbb{N} \approx \mathbb{W}$. (c) $\mathbb{N} \approx \mathbb{Z}$.

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4.3 The Rationals

Example. $\mathbb{N} \approx \mathbb{Q}$.

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4.4 The Reals

Example. $\mathbb{Q} \prec \mathbb{R}$.

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4.5 The Complex

Example. $\mathbb{R} \approx \mathbb{C}$.

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4.6 Set Levels

Definition. (a) A finite set is a set that is equinumerous to $\{1, 2, \cdots, n\}$ for some n, or it is empty. (b) A denumerable set is a set that is equinumerous to $\mathbb{N}$. A countable set is a set that is either finite or denumerable.

5 Cardinal Numbers

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6 Ordered Sets

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7 Axiomization

A precise development of Set Theory requires some axioms, which can be viewed as a start of the theory. Every proposition is duducted from either axioms or from lower-leveled propositions.

We always "let a set", and do something on it without verifying its existence. This would be a danger. For example.

(1) Let $S := \{x : x \notin x\}$.

(2) Let $r = \sum_{k=0}^{\infty} 2^k$.

So we ought to quote some basic, intuitive, and reasonable statements as what are called axioms.

The reason by which we choose these statements as axioms is because that they seems to be required and will not self-contradict. However, whether the seeming is real is a study in mathematical logic.

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