Report in Probability

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Recall that the **characteristic function** of a given random variable X is given by

$$\varphi(t) := Ee^{itX} = E\cos tX + iE\sin tX.$$

with several properties about its values. An important theorem called **Inversion Formula** says that:

Proposition 1 (Theorem 3.3.4.). *Let $\varphi(t) = \int e^{itx}\mu(dx)$ where μ is a probability measure. If $a < b$, then*

$$\lim_{T\to\infty}(2\pi)^{-1}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\varphi(t)dt = \mu(a,b)+\frac{1}{2}\mu(\{a,b\}).$$

We shall use this formula to show that

Theorem. *Let X be a random variable. If $\int|\varphi(t)|dt < \infty$, then*

(1) μ has a density. (in fact it is bounded continuous and is expressed by $f(y) = \frac{1}{2\pi}\int e^{-ity}\varphi(t)dt$.)

(2) Then $\varphi(t) \to 0$ as $t \to \infty$,

(3) and then μ has no point masses. i.e. there is no x such that $\mu(\{x\}) \neq 0$.

Moreover, none of the inverse of (1), (2), (3) is necessarily true.

Proof of (1). This follows from textbook, theorem 3.3.5. $\int_a^b e^{-ity}dy = \frac{e^{-ita}-e^{-itb}}{it}$ and absolute property imply that

$$\left|\frac{e^{-ita}-e^{-itb}}{it}\right| = \left|\int_a^b e^{-ity}dy\right| \le |b-a|.$$

Since, in fact, $\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\varphi(t)dt$ and $b-a$ are real,

$$\mu(a,b)+\frac{1}{2}\mu(\{a,b\}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ita}-e^{-itb}}{it}\varphi(t)dt \le \frac{b-a}{2\pi}\int_{-\infty}^{\infty}|\varphi(t)|dt.$$

I claim that μ has no point masses, for if a is a point mass, then

$$0 < \frac{1}{2}\mu(\{a\}) \le \frac{b-a}{2\pi}\int_{-\infty}^{\infty}|\varphi(t)|dt.$$

Let $b \to a$, then a is not a point mass, a contradiction. Thus by inversion formula and Fubini's theorem,

$$\begin{aligned}\mu(x, x+h) &= \frac{1}{2\pi}\int \frac{e^{-itx} - e^{-it(x+h)}}{it}\varphi(t)dt \\ &= \frac{1}{2\pi}\int \left(\int_x^{x+h} e^{-ity}dy\right)\varphi(t)dt = \int_x^{x+h}\left(\frac{1}{2\pi}\int e^{-ity}\varphi(t)dt\right)dy.\end{aligned}$$

So $f(y) = \frac{1}{2\pi}\int e^{-ity}\varphi(t)dt$ is the distribution. On the other hand, it converse fails. If we consider the **Exponential distribution** (example 3.3.6), who has density e^{-x} and ch.f. $\frac{1}{1-it}$, but the integral

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{1+t^2}}dt = \sinh^{-1} t\Big|_{-\infty}^{\infty} = \infty$$

is infinity. □

PROOF OF (2). We recall Riemann-Lebesgue Lemma that

Lemma 1. *If g is integrable then*

$$\lim_{\lambda\to\infty}\int g(x)\cos\lambda x dx = 0,$$
$$\lim_{\lambda\to\infty}\int g(x)\sin\lambda x dx = 0.$$

PROOF OF LEMMA. Since every function $g(x)$ can be written as the limit of a simple function sequence $g_n(x)$, where the integration of their difference can be less than an arbitrarily given $\varepsilon > 0$. That is, $\int |g(x) - g_N(x)|dx < \varepsilon$ (Zygmund pp.54) .

$$\begin{aligned}\left|\int g(x)\cos nx dx\right| &\le \left|\int g_N(x)\cos nx dx\right| + \int |g(x) - g_N(x)|dx \\ &\le \frac{2K}{n} + \varepsilon,\end{aligned}$$

so if $n >> 0$ we can conclude that $\int g(x)\cos nx dx = 0$. □

Then consider

$$\varphi(t) = \int (\cos(tX) + i\sin(tX))\, d\mu.$$

By above lemma, as $t \to \infty$, $\varphi(t) \to 0$. Its converse also fails. Let $\Omega = [0, 1]$ with Lebesgue measure. Let

$$X(\omega) = \begin{cases}\omega & \text{as } 0 \le \omega \le 0.5 \\ 2\omega - 0.5 & \text{as } 0.5 \le \omega \le 1.\end{cases}$$

Then the distribution function $F(x)$ has a non-differentiable point at $\frac{1}{2}$ and so cannot be expressed by $\int_{-\infty}^{x} f(y)dy$ for some f. Meanwhile,

$$\begin{aligned}\varphi(t) = Ee^{-itX} &= \int_0^{\frac{1}{2}} e^{-it\omega} d\omega + \int_{\frac{1}{2}}^{1} e^{-it(2\omega-0.5)} d\omega \\ &= \left(\frac{e^{-it\omega}}{-it}\right)\Big|_0^{\frac{1}{2}} + \left(\frac{e^{-it(2\omega-0.5)}}{-2it}\right)\Big|_{\frac{1}{2}}^{1} \to 0\end{aligned}$$

as $t \to \infty$. □

Proof of (3). We need the following properties.

Lemma 2. *(i)* $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)dt$.

(ii) If $P(X \in hZ) = 1$ *where* $h > 0$, *then its ch.f. has* $\varphi(\frac{2\pi}{h} + t) = \varphi(t)$, *so for* $x \in hZ$,

$$P(X = x) = \frac{h}{2\pi}\int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{-itx}\varphi(t)dt$$

(iii) Write $X = Y + b$, *then* $Ee^{itX} = e^{itb}Ee^{itY}$. *So if* $P(X \in b + hZ) = 1$, *the inversion formula in (ii) is alid for* $x \in b + hZ$.

Proposition 2. *If* X *and* Y *are independent and have ch.f.* φ *and distribution* μ. *Then*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2 dt = P(X - Y = 0) = \sum_x \mu(\{x\})^2.$$

To prove this proposition, we need the following.

Proposition 3. *(i) If* X *and* Y *are independent with distributions* μ *and* ν *then* $P(X + Y = 0) = \sum_y \mu(\{-y\})\nu(\{y\})$. *(ii) If* X *has continuous distribution then* $P(X = Y) = 0$.

Proof of original proposition. Then $X - Y$ has ch.f $\varphi \cdot \bar{\varphi} = |\varphi|^2$. Let $a = 0$ in (i) of the last theorem, then

$$P(X - Y = 0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2 dt.$$

By exercise 2.1.8, $P(X - Y = 0) = \sum_x \mu(\{x\})^2$. □

Back to its proof. We show that $\frac{1}{T}\int_0^T |\varphi(t)|^2 dt \to 0$ as $T \to \infty$. Given any $\varepsilon > 0$, by condition there is an $M > 0$ such that $|\varphi(t)| < \sqrt{\varepsilon}$ for any $T > M$. At this time,

$$\begin{aligned}0 \le \frac{1}{T}\int_0^T |\varphi(t)|^2 dt &= \frac{1}{T}\int_0^M |\varphi(t)|^2 dt + \int_M^T |\varphi(t)|^2 dt \\ &\le \frac{1}{T}\int_0^M |\varphi(t)|^2 dt + \int_M^T \varepsilon dt \le \frac{1}{T}\int_0^M |\varphi(t)|^2 dt + \varepsilon.\end{aligned}$$

Then $0 \leq \liminf_{T\to\infty} \frac{1}{T}\int_0^M |\varphi(t)|^2 dt + \varepsilon \leq \limsup_{T\to\infty} \frac{1}{T}\int_0^M |\varphi(t)|^2 dt + \varepsilon = \varepsilon$. Since ε is arbitrary, $\lim_{T\to\infty} \frac{1}{T}\int_0^T |\varphi(t)|^2 dt = 0$. Hence by last theorem, there is no point masses. The converse is false. Give a random variable $P(X = 0) = P(X = 0.5) = 0.5$. Then we claim that X has ch.f.

$$\varphi(t) = \prod_{j=1}^{\infty} \frac{1 + e^{-it2\cdot 3^{-j}}}{2}.$$

Since $e^{2\pi j} = 1$ for all $j \in \mathbb{N}$,

$$\varphi(3^k\pi) = \prod_{j=1}^{\infty} \frac{1 + e^{i2\pi\cdot 3^{k-j}}}{2} = \prod_{r=1}^{\infty} \frac{1 + e^{i2\pi\cdot 3^{-r}}}{2} = \varphi\pi.$$

Since $\varphi(\pi) \neq 0$, $\lim_{t\to\infty} \varphi(t) \neq 0$. $\square$

Proof of Claim. It follows from the lemma which states that if $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$, and φ_j is the ch.f. of X_j and $S_n \to S_\infty$ a.s. then S_∞ has ch.f. $\prod_{j=1}^{\infty} \varphi_j(t)$.

The proof is like this: Firstly we show that if $X_n \to X$ in probability and then $X_n \Rightarrow X$. Supposely that $X_n \to X$ in probability, then choose a bounded continuous g. Then $Eg(X_n) \to Eg(X)$ by BCT. Since g is arbitrary, by theorem $X_n \Rightarrow X$.

By basic properties of ch.f. S_n has ch.f. $\prod_{j=1}^{n} \varphi_j(t)$. By condition plus the last paragraph $S_n \Rightarrow S_\infty$. By continuity theorem that $\prod_{j=1}^{\infty} \varphi_(t) \to \varphi$, which is the ch.f. of S_∞. $\square$

References

[1] Rick Durrett, "Probability : theory and examples", Cambridge University Press, 2010.