A limit — By an elementary way

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It's easy to show that the series $\sum \frac{1}{n^2}$ converges, for that

$$\begin{aligned}\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...&<\frac{1}{1\times 1}+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...\\&=\frac{1}{1}+\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+...\end{aligned}$$

However, finding its limit require lots of efforts. Previously, great mathematician Euler evaluated the limit, $\frac{\pi^2}{6}$ through a unprecise method. Despite its unpreciseness, we still believe the truth of its limit —— its n-term values seems to approach it.

There're three interesting proofs: One is by complex variables; second is an application of Fourier Series; third is an elementary proof: We only use basic mathematics (those learned in high school) and squeeze theorem. I will restate the third proof here.

What is not only astonishing but incredible is that, the proof contains the following methods, which seem to be unrelated:

(1) Binomial Theorem.

(2) De Moivre Theorem.

(3) The relation between roots and coefficients.

(4) $\sin\theta<\theta<\tan\theta$.

(5) $1+\cot^2\theta=\csc^2\theta$.

(6) Squeeze Theorem.

It is really hard for most of people to think of them, and even use them. However, they work indeed, in critical parts. Now, expand the following

$$\begin{aligned}(\cos\theta+i\sin\theta)^{2n+1}&=\cos^{2n+1}\theta+C_1^{2n+1}\cos^{2n}(i\sin\theta)\\&\quad+C_2^{2n+1}\cos^{2n-1}(i\sin\theta)^2+...+C_n^n(i\sin\theta)^{2n+1}\\&=(\cos^{2n+1}\theta-C_2^{2n+1}\cos^{2n-1}\theta\sin^2\theta+...+\_\_\_\_)\\&\quad+i\cdot(C_1^{2n+1}\cos^{2n}(\sin\theta)-C_3^{2n+1}\cos^{2n-2}(\sin\theta)^3+...+\_\_\_)\end{aligned}$$

where

$$\theta=\frac{k\pi}{2n+1},\qquad k=1,2,3,...,n.$$

Moreover, for these θ's , De Moivre Theorem tells us that

$$\begin{aligned}(\cos\theta + i\sin\theta)^{2n+1} &= (\cos\frac{k\pi}{2n+1} + i\sin\frac{k\pi}{2n+1})^{2n+1}\\ &= \cos k\pi + i\sin k\pi \in \mathbb{R}.\end{aligned}$$

(which means, the imaginary parts must be 0.) Hence we find that

$$C_1^{2n+1}\cos^{2n}(\sin\theta) - C_3^{2n+1}\cos^{2n-2}(\sin\theta)^3 + ... + \_\_\_ = 0$$

Because $\theta = \frac{k\pi}{2n+1} \in (0, \frac{\pi}{2})$, $\sin^{2n+1}\theta \neq 0$. We can divid both sides of the last formula by $\sin^{2n+1}\theta$, getting

$$C_1^{2n+1}\cot^{2n} - C_3^{2n+1}\cot^{2n-2} + C_5^{2n+1}\cot^{2n-4} + ... + \_\_\_ = 0$$

Obverse the following equation:

$$C_1^{2n+1}Y^n - C_3^{2n+1}Y^{n-1} + C_5^{2n+1}Y^{n-2} + ... + \_\_\_ = 0$$

We know that it has at most n distinct roots. However, according to the previous result, we've indeed found out n roots:

$$\cot^2\frac{k\pi}{2n+1}, \qquad k = 1, 2, 3, ..., n.$$

This means, we know that the sum of these n roots

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} = \cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + ... + \cot^2\frac{n\pi}{2n+1} = \frac{C_3^{2n+1}}{C_1^{2n+1}} = \frac{(2n)(2n-1)}{6}.$$

Now, we see that, we have "made" the form of the sum of squares , π, and the number 6 in the denominator of the limit. Now, the π's has to get out of the cot's. Use the formula:

$$\sin\theta < \theta < \tan\theta, \qquad \text{while } \theta \in (0, \frac{\pi}{2}).$$

we find that

$$1 + \cot^2\theta = \csc^2\theta > \frac{1}{\theta^2} > \cot^2\theta$$

substituting $\frac{k\pi}{2n+1}$ for θ, and finding the sum where k goes from 1 to n, we get

$$n + \sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} > \sum_{k=1}^{n}\frac{1}{(\frac{k\pi}{2n+1})^2} > \sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1}$$

We rewrite it by

$$\frac{\pi^2}{(2n+1)^2}(n + \frac{(2n)(2n-1)}{6}) > \sum_{k=1}^{n}\frac{1}{k^2} > \frac{\pi^2}{(2n+1)^2}(\frac{(2n)(2n-1)}{6})$$

At last, by squeeze theorem, we will get the limit.

{ [Note] This is a partial rewrite of an article of the website:

http://episte.math.ntu.edu.tw

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