Natural numbers

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To begin construction on the system of natural numbers, there're some essential notions required, which are: 1 (one), natural numbers, $\cdot'$ (the next number of $\cdot$, the successor of $\cdot$), sets, and $=$ (identity). We agree that we're all familiar with properties of sets. Moreover, the notion of identity comes from logic, and satisfies:

$$\begin{array}{lll}
(R) & x = x & \forall x \\
(S) & x = y \Rightarrow y = x & \forall x, y \\
(T) & \text{if } x = y \text{ and } y = z, \text{ then } x = z & \forall x, y, z
\end{array}$$

We do not discuss what 1, a natural number, $\cdot'$ do really mean in detial. We only care about the properties of them from cognitive intuition. They are, in fact, called *axioms*, as follow: (Variables always mean natural number in this article.)

(I) 1 is a natural number.

(II) if $x = y$, then $x' = y'$.

(III) $\forall x$, $x' \neq 1$.

(IV) If $x' = y'$ then $x = y$.

(V) Let $\mathfrak{m}$ be a set of natural numbers and satisfies

  (a) $1 \in \mathfrak{m}$

  (b) $\forall x \in \mathfrak{m}$, $x' \in \mathfrak{m}$.

  Then $\mathfrak{m}$ contains all natural numbers $(*)$.

We denote $\mathbf{N}$ the set of all natural numbers. Hence, $(*)$ means $\mathfrak{m} = \mathbf{N}$.
Next, we define:

$$\begin{array}{lllll}
2 = 1' & 3 = 2' & 4 = 3' & 5 = 4' & 6 = 5' \\
7 = 6' & 8 = 7' & 9 = 8' & \mathbf{T} = 9' &
\end{array}$$

With them, we can exhibit additional operation on natural number. The following is its standard definition.

(i) $x + 1 = x'$.

(ii) $x + y' = (x + y)'$.

According to this, it's not hard to reach some *great* theorems:

Theorem 1. *(i)* $1 + 1 = 2$. *(ii)* $2 + 3 = 5$.

Proof. By definition, $1 + 1 = 1'$ and $2 = 1'$. By (S) and (T) we deduce $1 + 1 = 2$. Hence (i) is done.

By definition, $2 + 3 = 2 + 2' = (2 + 2)'$. Again $2 + 2 = 2 + 1' = (2 + 1)'$, and still again $2 + 1 = 2' = 3$. By (II) we get $(2 + 1)' = 3' = 4$, which means 2+2=4. Again $(2 + 2)' = 4' = 5$, which means $2 + 3 = 5$. □

Through the process of the proof, we additionally get that

$$2 + 1 = 3, \qquad 2 + 2 = 4.$$

[note]

We now may create an addition table for those pairs of numbers who are not too "large", for if we hope to evalute $5 + 7$, by similar argument as the above theorem, we might find that we have not yet possessed a notation for the result. (in fact, $5 + 7 = 9'''$, but we can not write 12 now).

[note]

The system of decimal expression and addition on decimals of natural numbers will be discuss far later.

We now proceed to show the so-called commutative law and associative law of natural numbers.

Theorem 2. *For all* y, $1 + y = y + 1$.

Proof. We want to prove it by (V). Let

$$\mathfrak{m} = \{y : \quad 1 + y = y + 1\}.$$

By (R), $1 + 1 = 1 + 1$, so $1 \in \mathfrak{m}$. Assume that $k \in \mathfrak{m}$, and we hope to reach the case that $k' \in \mathfrak{m}$. Since $1 + k = k + 1$, we evaluate

$$1 + k' = (1 + k)' = (k + 1)' = (k')' = k' + 1.$$

Hence $k' \in \mathfrak{m}$. Therefore $m = \mathbf{N}$, which ends the proof. □

Theorem 3. *For all* x, y, $x + y = y + x$.

Proof. Let

$$\mathfrak{m} = \{x : x + y = y + x \quad \forall y\}.$$

By above theorem, we have that $1 + y = y + 1 \quad \forall y$, so $1 \in \mathfrak{m}$. Assume that $k \in \mathfrak{m}$, namely,

$$k + y = y + k \qquad \forall y.$$

Then, we hope to show that $k' + y = y + k' \quad \forall y$. Denote

$$\mathfrak{n} = \{y : k' + y = y + k'\}.$$

Then we know that $1 \in \mathfrak{n}$. Assume that $t \in \mathfrak{n}$, namely

$$k' + t = t + k'.$$

We hope to show $k' + t' = t' + k'$. We do it as follow:

$$\begin{aligned} k' + t' &= (k' + t)' = (t + k')' = (t + k)'' \\ &= (k + t)'' = (k + t')' = (t' + k)' = t' + k'. \end{aligned}$$

Hence $t' \in \mathfrak{n}$. This means $\mathfrak{n} = \mathbf{N}$, i.e.

$$\forall y \qquad k' + y = y + k'.$$

We find immediately that $k' \in \mathfrak{m}$, and therefore $\mathfrak{m} = \mathbf{N}$. It is proved. □

Theorem 4. $(x + y) + z = x + (y + z) \quad \forall x, y, z.$

[hint] Let $\mathfrak{m} = \{z : (x + y) + z = x + (y + z) \quad \forall x \forall y\}$.

Theorem 5. *Given x, y, exactly one of the following holds:*

$$\begin{aligned} &x = y + u \qquad \exists u \\ &x = y \\ &y = x + v \qquad \exists v \end{aligned}$$

[hint] Let $\mathfrak{m} = \{x :$ The statement holds for all y $\}$. Then for fixed x, set $\mathfrak{n} = \{y :$ The statement holds $\}$. (V) gives a proof.

{ To be continued. This article will be updated later!! }