The sequence of n-th root of n

Wang-Shiuan Pahngerei

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This is an elementary way for showing that the sequence $\sqrt[n]{n}$ decreases and converges to 1.

1. *Show that* $\frac{(n+1)^n}{n^n}$<3

Proof. Binomial theorem tells us that

$$\begin{aligned}
\frac{(n+1)^n}{n^n} &= \frac{n^n + \binom{n}{1}n^{n-1} + \binom{n}{2}n^{n-2} + \binom{n}{3}n^{n-3} + \cdots + \binom{n}{n}1}{n^n} \\
&= \frac{n^n + \frac{n}{1}n^{n-1} + \frac{n\cdot(n-1)}{2\cdot 1}n^{n-2} + \frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}n^{n-3} + \cdots + \frac{n\cdot(n-1)\cdots 1}{n\cdot(n-1)\cdots 1}1}{n^n} \\
&\leq \frac{n^n + \frac{n}{1}n^{n-1} + \frac{n\cdot n}{2\cdot 1}n^{n-2} + \frac{n\cdot n\cdot n}{3\cdot 2\cdot 1}n^{n-3} + \cdots + \frac{n\cdot n\cdots n}{n\cdot(n-1)\cdots 1}1}{n^n} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!} \\
&\leq 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n} \\
&= 1 + 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\
&= 3 - \frac{1}{n} < 3
\end{aligned}$$

□

2. *Show that it decreases after the 5th term.*

Proof. Consider the quotient

$$\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = \left(\frac{(n+1)^n}{n^{n+1}}\right)^{\frac{1}{n\cdot(n+1)}} = \left(\frac{(n+1)^n}{n^n}\cdot\frac{1}{n}\right)^{\frac{1}{n\cdot(n+1)}} \leq \left(\frac{3}{n}\right)^{\frac{1}{n\cdot(n+1)}}$$

Now, if $\frac{3}{n}$<1, then $\left(\frac{3}{n}\right)^{\frac{1}{n\cdot(n+1)}}$ <1, and hence $\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}$<1. So while $n \geq 5$, we get that

$$\sqrt[n+1]{n+1} < \sqrt[n]{n}$$

□

3. *Show that it converges to 1.*

Proof. This is a proof rewritten from Homework I.5 of Calculus of the 15's.

We try to estimate the error, say $\sqrt[n]{n} = 1 + h_n$. Since

$$n = \left(\sqrt[n]{n}\right)^n = (1+h_n)^n = 1 + nh_n + \binom{n}{2}{h_n}^2 + \cdots + {h_n}^n$$
$$\geq 1 + \frac{n \cdot (n-1)}{2}{h_n}^2,$$

we get

$$0 \leq {h_n}^2 \leq \frac{2}{n} \to 0.$$

So $h_n \to 0$. This ends the proof. □

4. *As a test for how it gets close to 1, show that* $\sum_{k=1}^{\infty} {h_n}^j$ *diverges when* $j = 1$*, and converges when* $j = 2, 3, 4, \cdots$.

Proof. For $n \geq 3$,

$$(1 + \frac{1}{n})^n = \frac{(n+1)^n}{n^n} < 3 \leq n.$$

So

$$h_n = \sqrt[n]{n} - 1 \geq \frac{1}{n}.$$

Since $\sum \frac{1}{n}$ diverges, the series $\sum h_n$ (, by comparason test,) must diverge as well. This finish the case $j = 1$.

For $j = 2$, by binomial theorem again, we obtain $n \geq \binom{n}{3}{h_n}^3$. So

$${h_n}^2 \leq \sqrt[3]{\frac{36}{(n-1)^2(n-2)^2}} \leq \frac{1000}{n^{\frac{4}{3}}}.$$

Since we know that $\sum \frac{1}{n^{\frac{4}{3}}}$ converges, we (by comparason test) conclude that $\sum {h_n}^2$ converges.

If $j > 2$, since $h_n \to 0$, it must occurs that $h_n < \frac{1}{2}$ when n sufficiently large. So

$${h_n}^j < {h_n}^2.$$

By what is proved above (and comparason test), we finish this case. □