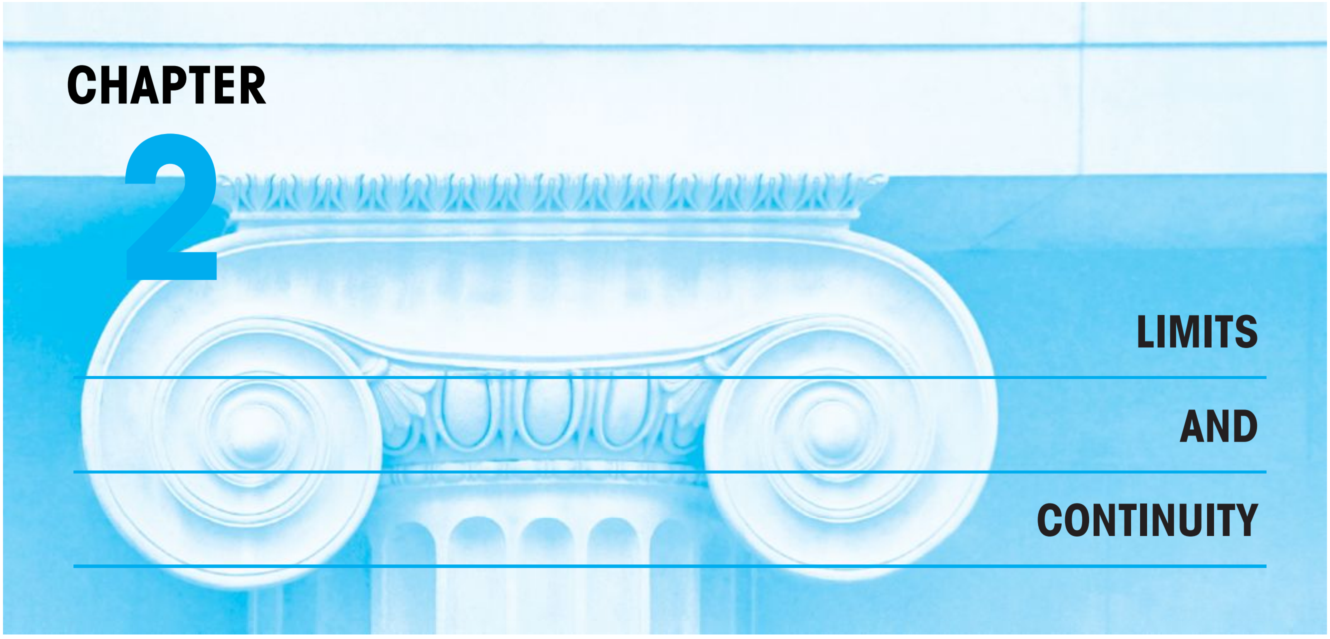


CHAPTER 2 LIMITS AND CONTINUITY

2.1 THE LIMIT PROCESS (AN INTUITIVE INTRODUCTION)

We could begin by saying that limits are important in calculus, but that would be a major understatement. *Without limits, calculus would not exist. Every single notion of calculus is a limit in one sense or another*. For example,

What is the slope of a curve? It is the limit of slopes of secant lines. (Figure 2.1.1.)

What is the length of a curve? It is the limit of the lengths of polygonal paths inscribed in the curve. (Figure 2.1.2)

What is the area of a region bounded by a curve? It is the limit of the sum of areas of approximating rectangles. (Figure 2.1.3)

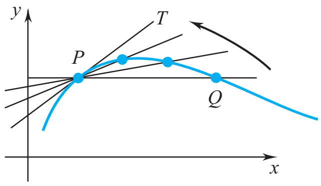


Figure 2.1.1

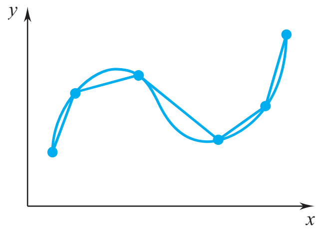


Figure 2.1.2

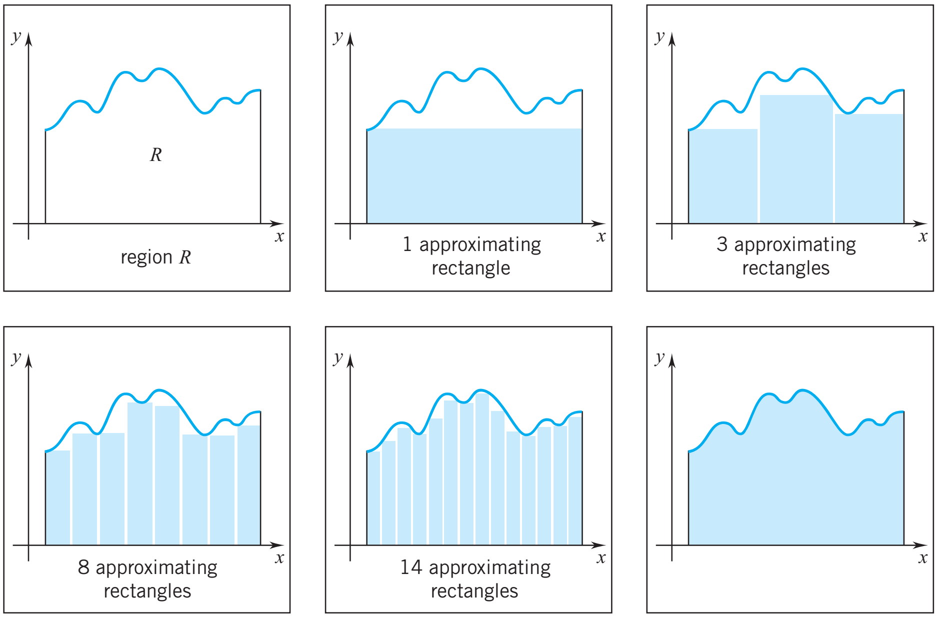


Figure 2.1.3

The Idea of Limit

Technically there are several limit processes, but they are all very similar. Once you master one of them, the others will pose few difficulties. The limit process that we start with is the one that leads to the notion of *continuity* and the notion of *differentiability*. At this stage our approach is completely informal. All we are trying to do here is lay an intuitive foundation for the mathematics that begins in Section 2.2

We start with a number c and a function f defined at all numbers x near c but not necessarily at c itself. In any case, whether or not f is defined at c and, if so, how is totally irrelevant.

Now let L be some real number. We say that *the limit of* $f(x)$ *as* x *tends to* c *is* L and write

$$\lim_{x \to c} f(x) = L$$

provided that (roughly speaking)

as x approaches c, f(x) approaches L

or (somewhat more precisely) provided that

$f(x)$ is close to L for all $x \neq c$ which are close to c.

Let's look at a few functions and try to apply this limit idea. Remember, our work at this stage is entirely intuitive.

Example 1 Set $f(x) = 4x + 5$ and take $c = 2$. As x approaches 2, $4x$ approaches 8 and $4x + 5$ approaches $8 + 5 = 13$. We conclude that

$$\lim_{x \to 2} f(x) = 13. \quad \square$$

Example 2 Set $f(x) = \sqrt{1-x}$ and take $c = -8$. As x approaches -8, $1 - x$ approaches 9 and $\sqrt{1-x}$ approaches 3. We conclude that

$$\lim_{x \to -8} f(x) = 3.$$

If for that same function we try to calculate

$$\lim_{x \to 2} f(x),$$

we run into a problem. The function $f(x) = \sqrt{1-x}$ is defined only for $x \leq 1$. It is therefore not defined for x near 2, and the idea of taking the limit as x approaches 2 makes no sense at all:

$$\lim_{x \to 2} f(x) \qquad \text{does not exist.} \quad \square$$

Example 3

$$\lim_{x \to 3} \frac{x^3 - 2x + 4}{x^2 + 1} = \frac{5}{2}.$$

First we work with the numerator: as x approaches 3, x^3 approaches 27, $-2x$ approaches -6, and $x^3 - 2x + 4$ approaches $27 - 6 + 4 = 25$. Now for the denominator: as x approaches 3, $x^2 + 1$ approaches 10. The quotient (it would seem) approaches $25/10 = 5/2$. $\square$

The curve in Figure 2.1.4 represents the graph of a function f. The number c is on the x-axis and the limit L is on the y-axis. As x approaches c along the x-axis, $f(x)$ approaches L along the y-axis.

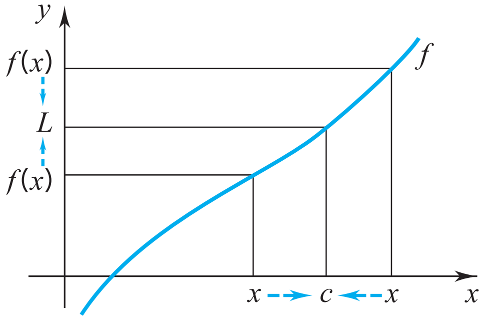


Figure 2.1.4

As we have tried to emphasize, in taking the limit of a function f as x tends to c, it does not matter whether f is defined at c and, if so, how it is defined there. The only thing that matters is the values taken on by f at numbers x near c. Take a look at the three cases depicted in Figure 2.1.5. In the first case, $f(c) = L$. In the second case, f is not defined at c. In the third case, f is defined at c, but $f(c) \neq L$. However, in each case

$$\lim_{x \to c} f(x) = L$$

because, as suggested in the figures,

as x approaches c, $f(x)$ approaches L.

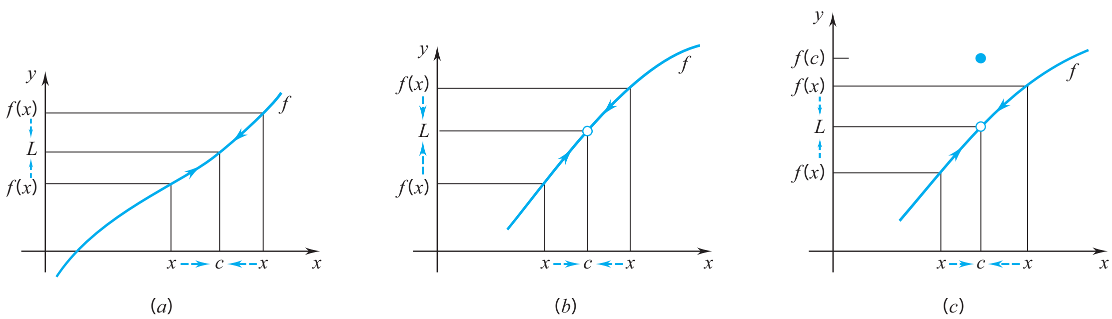


Figure 2.1.5

Example 4 Set $f(x) = \dfrac{x^2 - 9}{x - 3}$ and let $c = 3$. Note that the function f is not defined at 3: at 3, both numerator and denominator are 0. But that doesn't matter. For $x \neq 3$, and therefore *for all x near* 3,

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3.$$

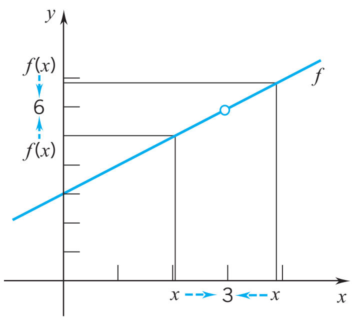


Figure 2.1.6

Therefore, if x is close to 3, then $\dfrac{x^2-9}{x-3} = x+3$ is close to $3+3=6$. We conclude that

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

The graph of f is shown in Figure 2.1.6. ❑

Example 5

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = 12.$$

The function $f(x) = \dfrac{x^3-8}{x-2}$ is undefined at $x=2$. But, as we said before, that doesn't matter. For all $x \neq 2$,

$$\frac{x^3-8}{x-2} = \frac{(x-2)(x^2+2x+4)}{x-2} = x^2+2x+4.$$

Therefore,

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12. \quad ❑$$

Example 6 If $f(x) = \begin{cases} 3x-4, & x \neq 0 \\ 10, & x = 0, \end{cases}$ then $\lim_{x\to 0} f(x) = -4.$

It does not matter that $f(0)=10$. For $x \neq 0$, and thus for all x near 0,

$$f(x) = 3x-4 \qquad \text{and therefore} \quad \lim_{x\to 0} f(x) = \lim_{x\to 0}(3x-4) = -4. \quad ❑$$

One-Sided Limits

Numbers x near c fall into two natural categories: those that lie to the left of c and those that lie to the right of c. We write

$$\lim_{x\to c^-} f(x) = L \qquad \text{[The left-hand limit of } f(x) \text{ as } x \text{ tends to } c \text{ is } L.]$$

to indicate that

as x approaches c from the left, f(x) approaches L.

We write

$$\lim_{x\to c^+} f(x) = L \qquad \text{[The right-hand limit of } f(x) \text{ as } x \text{ tends to } c \text{ is } L.]$$

to indicate that

as x approaches c from the right, f(x) approaches L.†

†The left-hand limit is sometimes written $\lim_{x\uparrow c} f(x)$ and the right-hand limit, $\lim_{x\downarrow c} f(x)$.

As an example, take the function indicated in Figure 2.1.7. As x approaches 5 from the left, $f(x)$ approaches 2; therefore

$$\lim_{x\to 5^-} f(x) = 2.$$

As x approaches 5 from the right, $f(x)$ approaches 4; therefore

$$\lim_{x\to 5^+} f(x) = 4.$$

The full limit, $\lim_{x\to 5} f(x)$, does not exist: consideration of $x < 5$ would force the limit to be 2, but consideration of $x > 5$ would force the limit to be 4.

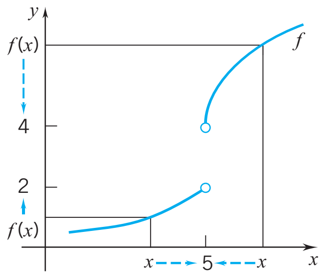


Figure 2.1.7

For a full limit to exist, both one-sided limits have to exist and they have to be equal.

Example 7 For the function f indicated in Figure 2.1.8,

$$\lim_{x\to(-2)^-} f(x) = 5 \qquad \text{and} \qquad \lim_{x\to(-2)^+} f(x) = 5.$$

In this case

$$\lim_{x\to -2} f(x) = 5.$$

It does not matter that $f(-2) = 3$.

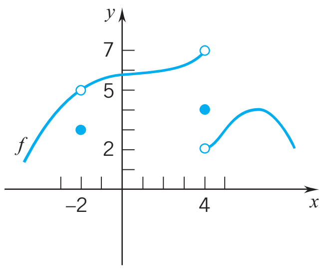


Figure 2.1.8

Examining the graph of f near $x = 4$, we find that

$$\lim_{x\to 4^-} f(x) = 7 \qquad \text{whereas} \qquad \lim_{x\to 4^+} f(x) = 2.$$

Since these one-sided limits are different,

$$\lim_{x\to 4} f(x) \qquad \text{does not exist.}$$ ❑

Example 8 Set $f(x) = x/|x|$. Note that $f(x) = 1$ for $x > 0$, and $f(x) = -1$ for $x < 0$:

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases} \qquad \text{(Figure 2.1.9)}$$

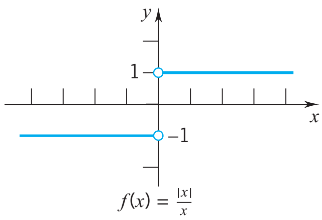


Figure 2.1.9

Let's try to apply the limit process at different numbers c.

If $c < 0$, then for all x sufficiently close to c, $x < 0$ and $f(x) = -1$. It follows that for $c < 0$

$$\lim_{x\to c} f(x) = \lim_{x\to c} (-1) = -1.$$

If $c > 0$, then for all x sufficiently close to c, $x > 0$ and $f(x) = 1$. It follows that for $c < 0$

$$\lim_{x\to c} f(x) = \lim_{x\to c} (1) = 1.$$

However, the function has no limit as x tends to 0:

$$\lim_{x\to 0^-} f(x) = -1 \qquad \text{but} \qquad \lim_{x\to 0^+} f(x) = 1.$$ ❑

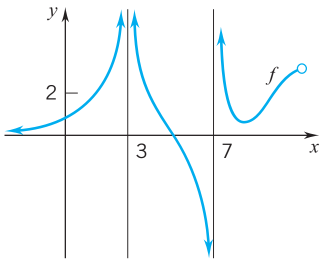


Figure 2.1.10

Example 9 We refer to the function indicated in Figure 2.1.10 and examine the behavior of $f(x)$ for x close to 3 and x close to 7.

As x approaches 3 from the left or from the right, $f(x)$ becomes arbitrarily large and cannot stay close to any number L. Therefore

$$\lim_{x\to 3} f(x) \qquad \text{does not exist.}$$

As x approaches 7 from the left, $f(x)$ becomes arbitrarily large negative and cannot stay close to any number L. Therefore

$$\lim_{x\to 7} f(x) \qquad \text{does not exist.}$$

The same conclusion can be reached by noting that as x approaches 7 from the right, $f(x)$ becomes arbitrarily large. ❑

Remark To indicate that $f(x)$ becomes arbitrarily large, we can write $f(x) \to \infty$. To indicate that $f(x)$ becomes arbitrarily large negative, we can write $f(x) \to -\infty$.

Go back to Figure 2.1.10, and note that for the function depicted there the following statements hold:

$$\text{as } x \to 3^-, \quad f(x) \to \infty \qquad \text{and} \qquad \text{as } x \to 3^+, \quad f(x) \to \infty.$$

Consequently,

$$\text{as } x \to 3, \qquad f(x) \to \infty.$$

Also,

$$\text{as } x \to 7^-, \quad f(x) \to -\infty \qquad \text{and} \qquad \text{as } x \to 7^+, \quad f(x) \to \infty.$$

We can therefore write

$$\text{as } x \to 7, \qquad |f(x)| \to \infty.$$ ❑

Example 10 We set

$$f(x) = \frac{1}{x-2}$$

and examine the behavior of $f(x)$ (a) as x tends to 4 and then (b) as x tends to 2.

(a) As x tends to 4, $x - 2$ tends to 2 and the quotient tends to 1/2. Thus

$$\lim_{x\to 4} f(x) = \frac{1}{2}.$$

(b) As x tends to 2 from the left, $f(x) \to -\infty$. (See Figure 2.1.11.) As x tends to 2 from the right, $f(x) \to \infty$. The function can have no numerical limit as x tends to 2. Thus

$$\lim_{x\to 2} f(x) \qquad \text{does not exist.}$$

However, it is true that

$$\text{as } x \to 2, \qquad |f(x)| \to \infty.$$ ❑

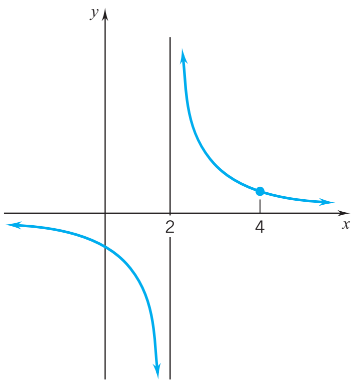


Figure 2.1.11

Example 11 Set $f(x) = \begin{cases} 1 - x^2, & x < 1 \\ 1/(x-1), & x > 1. \end{cases}$

For $x < 1$, $f(x) = 1 - x^2$. Thus

$$\lim_{x \to 1^-} f(x) = 0.$$

For $x > 1$, $f(x) = 1/(x-1)$. Therefore, as $x \to 1^+$, $f(x) \to \infty$. The function has no numerical limit as $x \to 1$:

$$\lim_{x \to 1} f(x) \qquad \text{does not exist.}$$

We now assert that

$$\lim_{x \to 1.5} f(x) = 2.$$

To see this, note that for x close to 1.5, $x > 1$ and therefore $f(x) = 1/(x-1)$. It follows that

$$\lim_{x \to 1.5} f(x) = \lim_{x \to 1.5} \frac{1}{x-1} = \frac{1}{0.5} = 2.$$

See Figure 2.1.12. ❑

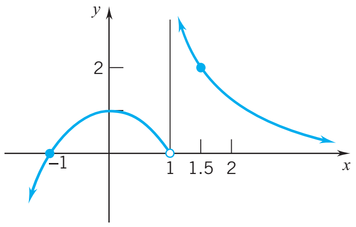


Figure 2.1.12

Example 12 Here we set $f(x) = \sin(\pi/x)$ and show that the function can have no limit as $x \to 0$.

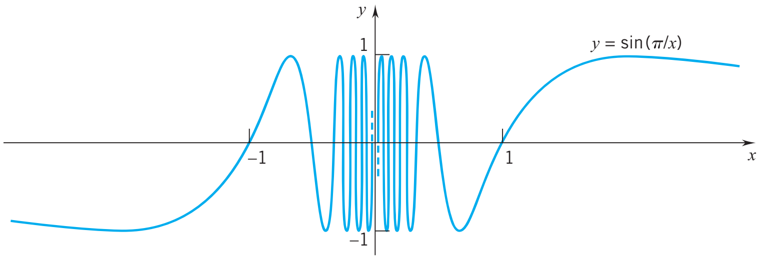


Figure 2.1.13

The function is not defined at $x = 0$, but, as you know, that's irrelevant. What keeps f from having a limit as $x \to 0$ is indicated in Figure 2.1.13. As $x \to 0$, $f(x)$ keeps oscillating between $y = 1$ and $y = -1$ and therefore cannot remain close to any one number L.† ❑

In our final example we rely on a calculator and deduce a limit from numerical calculation.

†We can approach $x = 0$

$$\text{by numbers } a_n = \frac{2}{4n+1} \qquad \text{and} \qquad \text{by numbers } b_n = -\frac{2}{4n+1},$$

$n = 0, 1, 2, 3, \ldots$. As you can check, $f(a_n) = 1$ and $f(b_n) = -1$. This confirms the oscillatory behavior of f near $x = 0$.

Example 13 Let $f(x) = (\sin x)/x$. If we try to evaluate f at 0, we get the meaningless ratio 0/0; f is not defined at $x = 0$. However, f is defined for all $x \neq 0$, and so we can consider

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

We select numbers that approach 0 closely from the left and numbers that approach 0 closely from the right. Using a calculator, we evaluate f at these numbers. The results are tabulated in Table 2.1.1.

Table 2.1.1

(Left side)		(Right side)	
x (radians)	$\frac{\sin x}{x}$	x (radians)	$\frac{\sin x}{x}$
−1	0.84147	1	0.84147
−0.5	0.95885	0.5	0.95885
−0.1	0.99833	0.1	0.99833
−0.01	0.99998	0.01	0.99998
−0.001	0.99999	0.001	0.99999

These calculations suggest that

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

and therefore that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The graph of f, shown in Figure 2.1.14, supports this conclusion. A proof that this limit is indeed 1 is given in Section 2.5. ❑

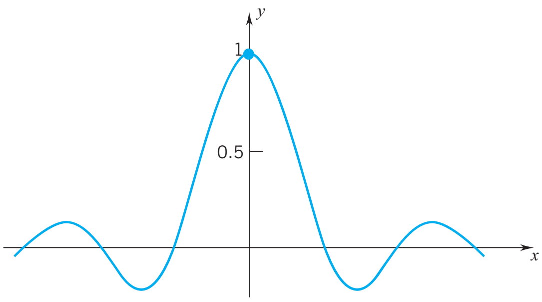


Figure 2.1.14

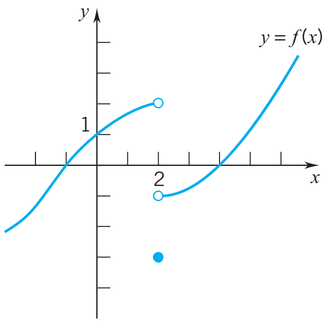
If you have found all this to be imprecise, you are absolutely right. Our work so far has been imprecise. In Section 2.2 we will work with limits in a more coherent manner.

EXERCISES 2.1

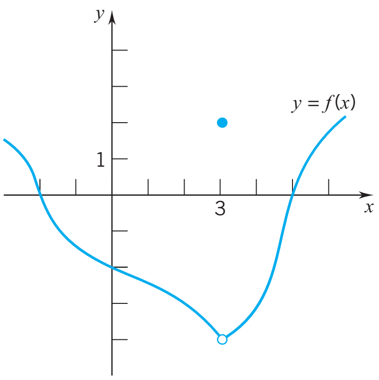
Exercises 1–10. You are given a number c and the graph of a function f. Use the graph to find

(a) $\lim_{x\to c^-} f(x)$ (b) $\lim_{x\to c^+} f(x)$ (c) $\lim_{x\to c} f(x)$ (d) $f(c)$

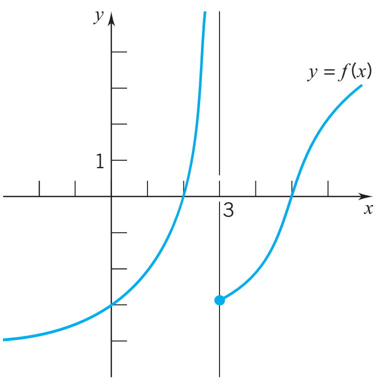
1. $c = 2$.



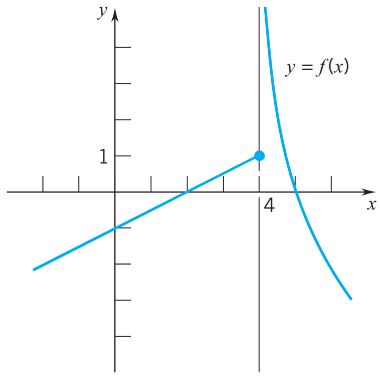
2. $c = 3$.



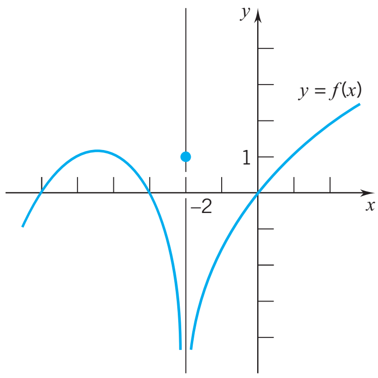
3. $c = 3$.



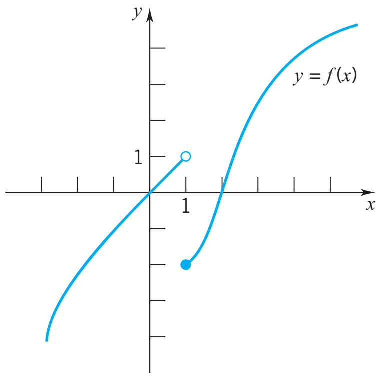
4. $c = 4$.



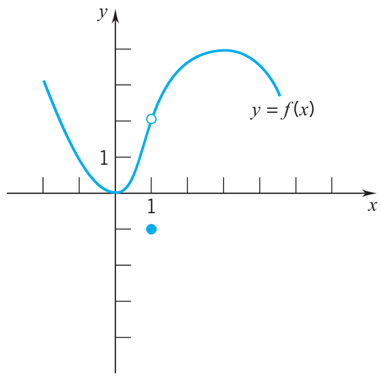
5. $c = -2$.



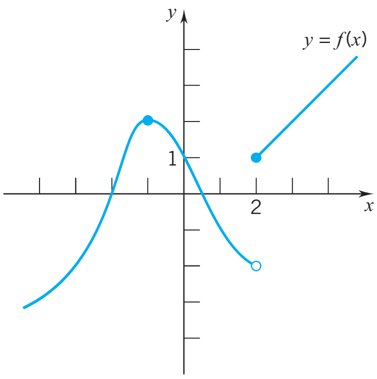
6. $c = 1$.



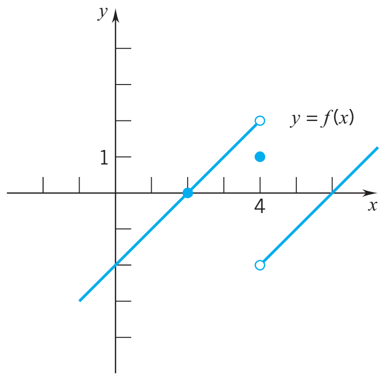
7. $c = 1$.



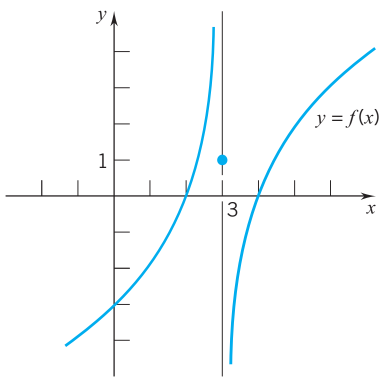
8. $c = -1$.



9. $c = 2$.

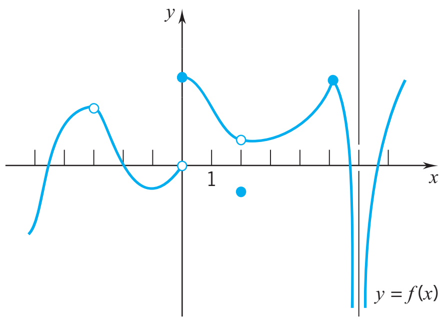


10. $c = 3$.

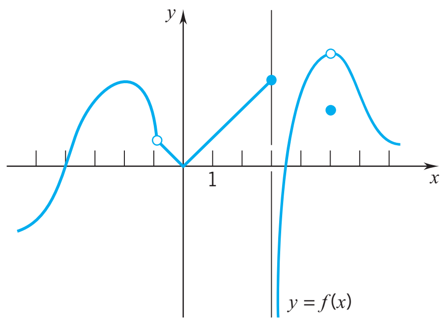


Exercises 11–12. Give the values of c for which $\lim_{x\to c} f(x)$ does not exist.

11.



12.



Exercises 13–49. Decide on intuitive grounds whether or not the indicated limit exists; evaluate the limit if it does exist.

13. $\lim_{x\to 0}(2x-1)$.

14. $\lim_{x\to 1}(2-5x)$.

15. $\lim_{x\to -2}(x^2-2x+4)$.

16. $\lim_{x\to 4}\sqrt{x^2+2x+1}$.

17. $\lim_{x\to -3}(|x|-2)$.

18. $\lim_{x\to 0}\dfrac{1}{|x|}$.

19. $\lim_{x\to 1}\dfrac{3}{x+1}$.

20. $\lim_{x\to -1}\dfrac{4}{x+1}$.

21. $\lim_{x\to -1}\dfrac{-2}{x+1}$.

22. $\lim_{x\to 2}\dfrac{1}{3x-6}$.

23. $\lim_{x\to 3}\dfrac{2x-6}{x-3}$.

24. $\lim_{x\to 3}\dfrac{x^2-6x+9}{x-3}$.

25. $\lim_{x\to 3}\dfrac{x-3}{x^2-6x+9}$.

26. $\lim_{x\to 2}\dfrac{x^2-3x+2}{x-2}$.

27. $\lim_{x\to 2}\dfrac{x-2}{x^2-3x+2}$.

28. $\lim_{x\to 1}\dfrac{x-2}{x^2-3x+2}$.

29. $\lim_{x\to 0}\left(x+\dfrac{1}{x}\right)$.

30. $\lim_{x\to 1}\left(x+\dfrac{1}{x}\right)$.

31. $\lim_{x\to 0}\dfrac{2x-5x^2}{x}$.

32. $\lim_{x\to 3}\dfrac{x-3}{6-2x}$.

33. $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

34. $\lim_{x\to 1}\dfrac{x^3-1}{x-1}$.

35. $\lim_{x\to 1}\dfrac{x^3-1}{x+1}$.

36. $\lim_{x\to 1}\dfrac{x^2+1}{x^2-1}$.

37. $\lim_{x\to 0} f(x)$; $\quad f(x) = \begin{cases} 1, & x \neq 0 \\ 3, & x = 0. \end{cases}$

38. $\lim_{x\to 1} f(x)$; $\quad f(x) = \begin{cases} 3x, & x < 1 \\ 3, & x > 1. \end{cases}$

39. $\lim_{x\to 4} f(x)$; $\quad f(x) = \begin{cases} x^2, & x \neq 4 \\ 0, & x = 4. \end{cases}$

40. $\lim_{x\to 0} f(x)$; $\quad f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x > 0. \end{cases}$

41. $\lim_{x\to 0} f(x)$; $\quad f(x) = \begin{cases} x^2, & x < 0 \\ 1+x, & x > 0. \end{cases}$

42. $\lim_{x\to 1} f(x)$; $\quad f(x) = \begin{cases} 2x, & x < 1 \\ x^2+1, & x > 1. \end{cases}$

43. $\lim_{x\to 2} f(x)$; $\quad f(x) = \begin{cases} 3x, & x < 1 \\ x+2, & x \geq 1. \end{cases}$

44. $\lim_{x\to 0} f(x)$; $\quad f(x) = \begin{cases} 2x, & x \leq 1 \\ x+1, & x > 1. \end{cases}$

45. $\lim_{x\to 0} f(x)$; $\quad f(x) = \begin{cases} 2, & x \text{ rational} \\ -2, & x \text{ irrational}. \end{cases}$

46. $\lim_{x\to 1} f(x)$; $\quad f(x) = \begin{cases} 2x, & x \text{ rational} \\ 2, & x \text{ irrational}. \end{cases}$

47. $\lim_{x\to 1} \dfrac{\sqrt{x^2+1}-\sqrt{2}}{x-1}$.

48. $\lim_{x\to 5} \dfrac{\sqrt{x^2+5}-\sqrt{30}}{x-5}$.

49. $\lim_{x\to 1} \dfrac{x^2+1}{\sqrt{2x+2}-2}$.

Exercises 50–54. After estimating the limit using the prescribed values of x, validate or improve your estimate by using a graphing utility.

50. Estimate

$$\lim_{x\to 0} \frac{1-\cos x}{x} \qquad \text{(radian measure)}$$

by evaluating the quotient at $x = \pm 1, \pm 0.1, \pm 0.01, \pm 0.001$.

51. Estimate

$$\lim_{x\to 0} \frac{\tan 2x}{x} \qquad \text{(radian measure)}$$

by evaluating the quotient at $x = \pm 1, \pm 0.1, \pm 0.01, \pm 0.001$.

52. Estimate

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} \qquad \text{(radian measure)}$$

after evaluating the quotient at $x = \pm 1, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$.

53. Estimate

$$\lim_{x\to 1} \frac{x^{3/2}-1}{x-1}$$

by evaluating the quotient at $x = 0.9, 0.99, 0.99, 0.9999$ and at $x = 1.1, 1.01, 1.001, 1.0001$.

54. Estimate

$$\lim_{x\to 0} \frac{2\cos x - 2 + x^2}{x^4} \qquad \text{(radian measure)}$$

by evaluating the quotient at $x = \pm 1, \pm 0.1, \pm 0.01, \pm 0.0001, \pm 0.0001$.

55. (a) Use a graphing utility to estimate $\lim_{x\to 4} f(x)$:

(i) $f(x) = \dfrac{2x^2-11x+12}{x-4}$;

(ii) $f(x) = \dfrac{2x^2-11x+12}{x^2-8x+16}$.

(b) Use a CAS to find each of the limits in part (a).

56. (a) Use a graphing utility to estimate $\lim_{x\to 4} f(x)$:

(i) $f(x) = \dfrac{3x^2-10x-8}{5x^2+16x-16}$;

(ii) $f(x) = \dfrac{5x^2-26x+24}{4x^2-11x-20}$.

(b) Use a CAS to find each of the limits in part (a).

57. (a) Use a graphing utility to estimate $\lim_{x\to 2} f(x)$:

(i) $f(x) = \dfrac{\sqrt{6-x}-x}{x-2}$; (ii) $f(x) = \dfrac{x^2-4x+4}{x-\sqrt{6-x}}$.

(b) Use a CAS to find each of the limits in part (a).

58. (a) Use a graphing utility to estimate $\lim_{x\to 2} f(x)$

(i) $f(x) = \dfrac{2x-\sqrt{18-x}}{4-x^2}$; (ii) $f(x) = \dfrac{2-\sqrt{2x}}{\sqrt{8x}-4}$.

(b) Use a CAS to find each of the limits in part (a).

Exercises 59–62. Use a graphing utility to find at least one number c at which $\lim_{x\to c} f(x)$ does not exist.

59. $f(x) = \dfrac{x+1}{|x^3+1|}$.

60. $f(x) = \dfrac{|6x^2-x-35|}{2x-5}$.

61. $f(x) = \dfrac{|x|}{x^5+2x^4+13x^3+26x^2+36x+72}$.

62. $f(x) = \dfrac{5x^3-22x^2+15x+18}{x^3-9x^2+27x-27}$.

63. Use a graphing utility to draw the graphs of

$$f(x) = \frac{1}{x}\sin x \qquad \text{and} \qquad g(x) = x\sin\left(\frac{1}{x}\right)$$

for $x \neq 0$ between $-\pi/2$ and $\pi/2$. Describe the behavior of $f(x)$ and $g(x)$ for x close to 0.

64. Use a graphing utility to draw the graphs of

$$f(x) = \frac{1}{x}\tan x \qquad \text{and} \qquad g(x) = x\tan\left(\frac{1}{x}\right)$$

for $x \neq 0$ between $-\pi/2$ and $\pi/2$. Describe the behavior of $f(x)$ and $g(x)$ for x close to 0.

2.2 DEFINITION OF LIMIT

In Section 2.1 we tried to give you an intuitive feeling for the limit process. However, our description was too vague to be called "mathematics." We relied on statements such as

"as x approaches c, $f(x)$ approaches L"

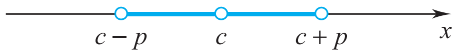
and

"$f(x)$ is close to L for all $x \neq c$ which are close to c."

But what exactly do these statements mean? What are we saying by stating that "$f(x)$ approaches L"? How close is close?

In this section we formulate the limit process in a coherent manner and, by so doing, establish a foundation for more advanced work.

As before, in taking the limit of $f(x)$ as x approaches c, we don't require that f be defined at c, but we do require that f be defined at least on an open interval $(c-p, c+p)$ except possibly at c itself.



To say that

$$\lim_{x \to c} f(x) = L$$

is to say that $|f(x) - L|$ can be made as small as we choose, *less than any* $\epsilon > 0$ *we choose*, by restricting x to a sufficiently small set of the form $(c-\delta, c) \cup (c, c+\delta)$, *by restricting* x *by an inequality of the form* $0 < |x-c| < \delta$ *with* $\delta > 0$ *sufficiently small.*

Phrasing this idea precisely, we have the following definition.

DEFINITION 2.2.1 THE LIMIT OF A FUNCTION

Let f be a function defined at least on an open interval $(c-p, c+p)$ except possibly at c itself. We say that

$$\lim_{x \to c} f(x) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

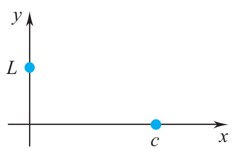


Figure 2.2.1

Figures 2.2.1 and 2.2.2 illustrate this definition.

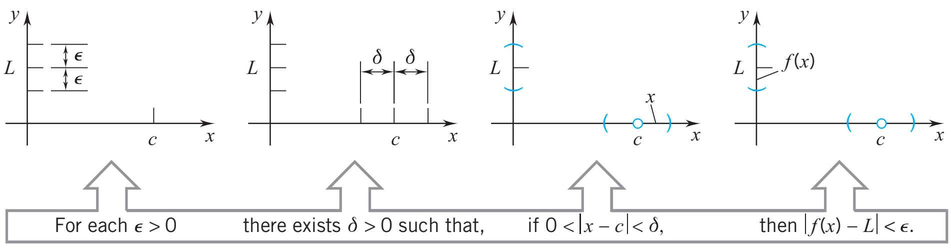


Figure 2.2.2

Except in the case of a constant function, the choice of δ depends on the previous choice of ϵ. We do not require that there exists a number δ which "works" for *all* ϵ, but rather, that for each ϵ there exists a δ which "works" for that particular ϵ.

In Figure 2.2.3, we give two choices of ϵ and for each we display a suitable δ. For a δ to be suitable, all points within δ of c (with the possible exception of c itself) must be taken by the function f to within ϵ of L. In part (b) of the figure, we began with a smaller ϵ and had to use a smaller δ.

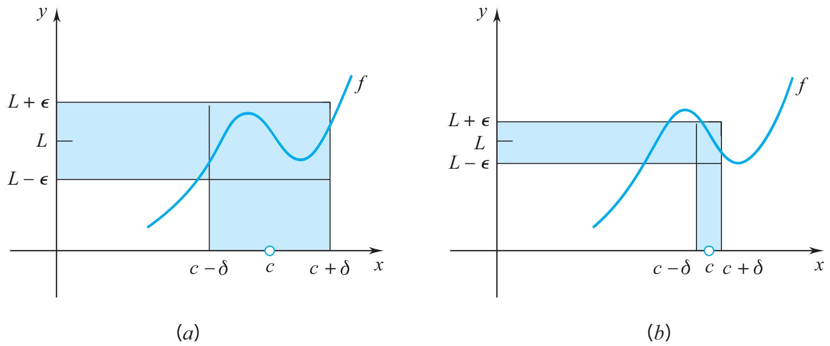


Figure 2.2.3

The δ of Figure 2.2.4 is too large for the given ϵ. In particular, the points marked x_1 and x_2 in the figure are not taken by f to within ϵ of L.

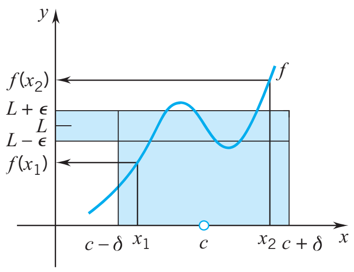


Figure 2.2.4

As these illustrations suggest, the limit process can be described entirely in terms of open intervals. (See Figure 2.2.5.)

(2.2.2)

> Let f be defined at least on an open interval $(c-p, c+p)$ except possibly at c itself. We say that
>
> $$\lim_{x\to c} f(x) = L$$
>
> if for each open interval $(L-\epsilon, L+\epsilon)$ there is an open interval $(c-\delta, c+\delta)$ such that all the numbers in $(c-\delta, c+\delta)$, with the possible exception of c itself, are mapped by f into $(L-\epsilon, L+\epsilon)$.

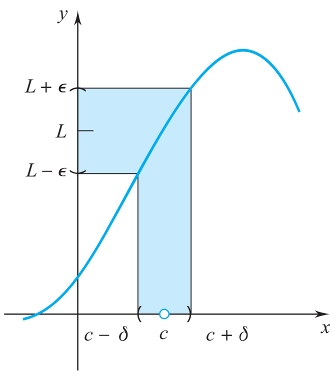


Figure 2.2.5

Next we apply the ϵ, δ definition of limit to a variety of functions. At first you may find the ϵ, δ arguments confusing. It usually takes a little while for the ϵ, δ idea to take hold.

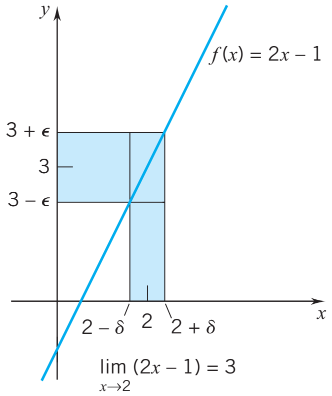


Figure 2.2.6

Example 1 Show that

$$\lim_{x \to 2}(2x - 1) = 3. \qquad \text{(Figure 2.2.6)}$$

Finding a δ. Let $\epsilon > 0$. We seek a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 2| < \delta, \quad \text{then} \quad |(2x - 1) - 3| < \epsilon.$$

What we have to do first is establish a connection between

$$|(2x - 1) - 3| \quad \text{and} \quad |x - 2|.$$

The connection is evident:

$$(*) \qquad |(2x - 1) - 3| = |2x - 4| = 2|x - 2|.$$

To make $|(2x - 1) - 3|$ less than ϵ, we need to make $2|x - 2| < \epsilon$, which we can accomplish by making $|x - 2| < \epsilon/2$. This suggests that we choose $\delta = \frac{1}{2}\epsilon$.

Showing that the δ *"works."* If $0 < |x - 2| < \frac{1}{2}\epsilon$, then $2|x - 2| < \epsilon$ and, by $(*)$, $|(2x - 1) - 3| < \epsilon$. ❑

Remark In Example 1 we chose $\delta = \frac{1}{2}\epsilon$, but we could have chosen *any* positive number δ less than $\frac{1}{2}\epsilon$. In general, if a certain δ^* "works" for a given ϵ, then any δ less than δ^* will also work. ❑

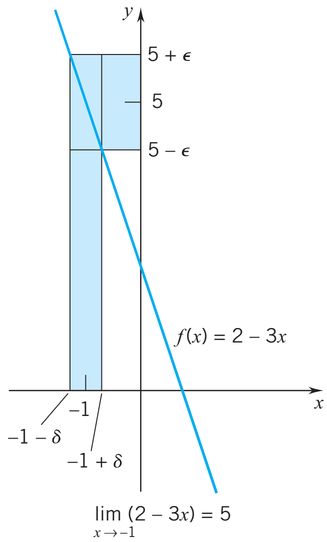


Figure 2.2.7

Example 2 Show that

$$\lim_{x \to -1}(2 - 3x) = 5. \qquad \text{(Figure 2.2.7)}$$

Finding a δ. Let $\epsilon > 0$. We seek a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - (-1)| < \delta, \quad \text{then} \quad |(2 - 3x) - 5| < \epsilon.$$

To find a connection between

$$|x - (-1)| \quad \text{and} \quad |(2 - 3x) - 5|,$$

we simplify both expressions:

$$|x - (-1)| = |x + 1|$$

and

$$|(2 - 3x) - 5| = |-3x - 3| = |-3||x + 1| = 3|x + 1|.$$

We can conclude that

$$(**) \qquad |(2 - 3x) - 5| = 3|x - (-1)|.$$

We can make the expression on the left less than ϵ by making $|x - (-1)|$ less than $\epsilon/3$. This suggests that we set $\delta = \frac{1}{3}\epsilon$.

Showing that the δ *"works."* If $0 < |x - (-1)| < \frac{1}{3}\epsilon$, then $3|x - (-1)| < \epsilon$ and, by $(**)$, $|(2 - 3x) - 5| < \epsilon$. ❑

Three Basic Limits

Here we apply the ϵ, δ method to confirm three basic limits that are intuitively obvious. (If the ϵ, δ method did not confirm these limits, then the method would have been thrown out a long time ago.)

Example 3 For each number c,

(2.2.3)
$$\lim_{x \to c} x = c.$$
(Figure 2.2.8)

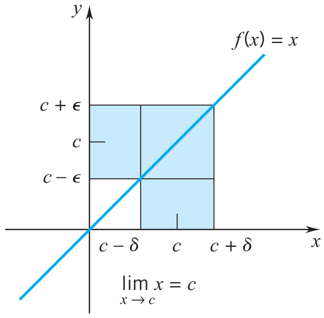


Figure 2.2.8

PROOF Let c be a real number and let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |x - c| < \epsilon.$$

Obviously we can choose $\delta = \epsilon$. ❑

Example 4 For each real number c

(2.2.4)
$$\lim_{x \to c} |x| = |c|.$$
(Figure 2.2.9)

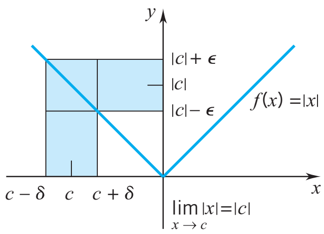


Figure 2.2.9

PROOF Let c be a real number and let $\epsilon > 0$. We seek a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad \big||x| - |c|\big| < \epsilon.$$

Since

$$\big||x| - |c|\big| \le |x - c|, \qquad [(1.3.7)]$$

we can choose $\delta = \epsilon$, for

$$\text{if} \quad 0 < |x - c| < \epsilon, \quad \text{then} \quad \big||x| - |c|\big| < \epsilon. \quad ❑$$

Example 5 For each constant k

(2.2.5)
$$\lim_{x \to c} k = k.$$
(Figure 2.2.10)

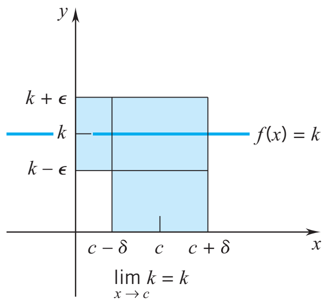


Figure 2.2.10

PROOF Here we are dealing with the constant function

$$f(x) = k.$$

Let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |k - k| < \epsilon.$$

Since $|k - k| = 0$, we always have

$$|k - k| < \epsilon$$

no matter how δ is chosen; in short, any positive number will do for δ. ❑

Usually ϵ, δ arguments are carried out in two stages. First we do a little scratch work, labeled "finding a δ" in Examples 1 and 2. This scratch work involves working backward from $|f(x) - L| < \epsilon$ to find a $\delta > 0$ sufficiently small so that we can begin with the inequality $0 < |x - c| < \delta$ and arrive at $|f(x) - L| < \epsilon$. This first stage is just preliminary, but it shows us how to proceed in the second stage. The second stage consists of showing that the δ "works" by verifying that, for our choice of δ, it is true that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

The next two examples will give you a better feeling for this idea of working backward to find a δ.

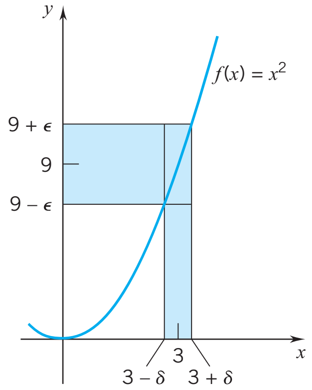


Figure 2.2.11

Example 6

$$\lim_{x \to 3} x^2 = 9 \qquad \text{(Figure 2.2.11)}$$

Finding a δ. Let $\epsilon > 0$. We seek a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 3| < \delta, \quad \text{then} \quad |x^2 - 9| < \epsilon.$$

The connection between $|x - 3|$ and $|x^2 - 9|$ can be found by factoring:

$$x^2 - 9 = (x + 3)(x - 3),$$

and thus,

$$(*) \qquad |x^2 - 9| = |x + 3||x - 3|.$$

At this point, we need to get an estimate for the size of $|x + 3|$ for x close to 3. For convenience, we'll take x within one unit of 3.

If $|x - 3| < 1$, then $2 < x < 4$ and

$$|x + 3| \le |x| + |3| = x + 3 < 7.$$

Therefore, by $(*)$,

$$(**) \qquad \text{if} \quad |x - 3| < 1, \quad \text{then} \quad |x^2 - 9| < 7|x - 3|.$$

If, in addition, $|x - 3| < \epsilon/7$, then it will follow that

$$|x^2 - 9| < 7(\epsilon/7) = \epsilon.$$

This means that we can let $\delta =$ the minimum of 1 and $\epsilon/7$.

Showing that the δ *"works."* Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/7\}$ and assume that

$$0 < |x - 3| < \delta.$$

Then

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \epsilon/7.$$

By $(**)$,

$$|x^2 - 9| < 7|x - 3|,$$

and since $|x - 3| < \epsilon/7$, we have

$$|x^2 - 9| < 7(\epsilon/7) = \epsilon. \qquad ❑$$

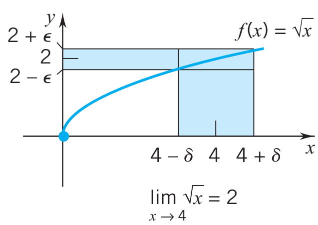


Figure 2.2.12

Example 7

$$\lim_{x \to 4} \sqrt{x} = 2. \qquad \text{(Figure 2.2.12)}$$

Finding a δ. Let $\epsilon > 0$. We seek a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 4| < \delta, \quad \text{then} \quad |\sqrt{x} - 2| < \epsilon.$$

To be able to form $\sqrt{x}$, we need to have $x \geq 0$. To ensure this, we must have $\delta \leq 4$. (Explain.)

Remembering that we must have $\delta \leq 4$, let's move on to find a connection between $|x-4|$ and $|\sqrt{x}-2|$. With $x \geq 0$, we can form $\sqrt{x}$ and write

$$x - 4 = (\sqrt{x})^2 - 2^2 = (\sqrt{x}+2)(\sqrt{x}-2).$$

Taking absolute values, we have

$$|x-4| = |\sqrt{x}+2||\sqrt{x}-2|.$$

Since $|\sqrt{x}+2| \geq 2 > 1$, it follows that

$$|\sqrt{x}-2| < |x-4|.$$

This last inequality suggests that we can simply set $\delta \leq \epsilon$. But remember the requirement $\delta \leq 4$. We can meet both requirements on δ by setting $\delta =$ the minimum of 4 and ϵ.

Showing that the δ "works." Let $\epsilon > 0$. Choose $\delta = \min\{4, \epsilon\}$ and assume that

$$0 < |x-4| < \delta.$$

Since $\delta \leq 4$, we have $x \geq 0$, and so $\sqrt{x}$ is defined. Now, as shown above,

$$|x-4| = |\sqrt{x}+2||\sqrt{x}-2|.$$

Since $|\sqrt{x}+2| \geq 2 > 1$, we can conclude that

$$|\sqrt{x}-2| < |x-4|.$$

Since $|x-4| < \delta$ and $\delta \leq \epsilon$, it does follow that $|x-2| < \epsilon$. ❑

There are several different ways of formulating the same limit statement. Sometimes one formulation is more convenient, sometimes another. In particular, it is useful to recognize that the following four statements are equivalent:

(2.2.6)

(i) $\lim_{x \to c} f(x) = L$	(ii) $\lim_{h \to 0} f(c+h) = L$
(iii) $\lim_{x \to c} (f(x) - L) = 0$	(iv) $\lim_{x \to c} \lvert f(x) - L \rvert = 0.$

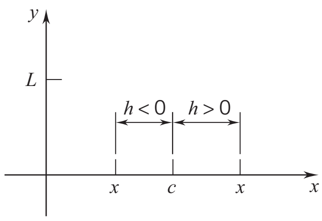


Figure 2.2.13

The equivalence of (i) and (ii) is illustrated in Figure 2.2.13: simply think of h as being the signed distance from c to x. Then $x = c + h$, and x approaches c iff h approaches 0. It is a good exercise in ϵ, δ technique to prove that (i) is equivalent to (ii).

Example 8 For $f(x) = x^2$, we have

$$\lim_{x \to 3} x^2 = 9 \qquad \lim_{h \to 0} (3+h)^2 = 9$$
$$\lim_{x \to 3} (x^2 - 9) = 0 \qquad \lim_{x \to 3} |x^2 - 9| = 0. \quad ❑$$

We come now to the ϵ, δ definitions of one-sided limits. These are just the usual ϵ, δ statements, except that for a left-hand limit, the δ has to "work" only for x to the left of c, and for a right-hand limit, the δ has to "work" only for x to the right of c.

DEFINITION 2.2.7 LEFT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c-p, c)$. We say that

$$\lim_{x\to c^-} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if} \quad c-\delta < x < c, \quad \text{then} \quad |f(x)-L| < \epsilon.$$

DEFINITION 2.2.8 RIGHT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c, c+p)$. We say that

$$\lim_{x\to c^+} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if} \quad c < x < c+\delta \quad \text{then} \quad |f(x)-L| < \epsilon.$$

As our intuitive approach in Section 2.1 suggested,

(2.2.9) $$\lim_{x\to c} f(x) = L \quad \text{iff} \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

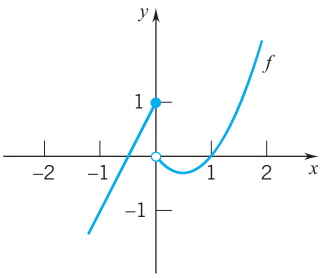


Figure 2.2.14

The result follows from the fact that any δ that "works" for the limit will work for both one-sided limits, and any δ that "works" for both one-sided limits will work for the limit.

Example 9 For the function defined by setting

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ x^2 - x, & x > 0, \end{cases} \qquad \text{(Figure 2.2.14)}$$

$\lim_{x\to 0} f(x)$ does not exist.

PROOF The left- and right-hand limits at 0 are as follows:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 1, \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2 - x) = 0.$$

Since these one-sided limits are different, $\lim_{x\to 0} f(x)$ does not exist. ❑

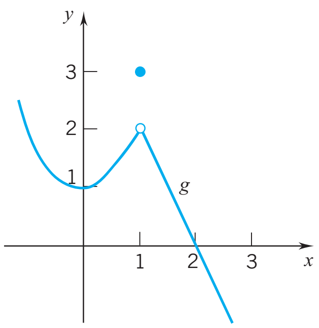


Figure 2.2.15

Example 10 For the function defined by setting

$$g(x) = \begin{cases} 1+x^2, & x < 1 \\ 3, & x = 1 \\ 4-2x, & x > 1, \end{cases} \qquad \text{(Figure 2.2.15)}$$

$\lim_{x\to 1} g(x) = 2.$

PROOF The left- and right-hand limits at 1 are as follows:

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (1 + x^2) = 2, \qquad \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (4 - 2x) = 2.$$

Thus, $\lim_{x\to 1} g(x) = 2$. NOTE: It does not matter that $g(1) \neq 2$. ❑

At an endpoint of the domain of a function we can't take a (full) limit and we can't take a one-sided limit from the side on which the function is not defined, but we can try to take a limit from the side on which the function is defined. For example, it makes no sense to write

$$\lim_{x\to 0} \sqrt{x} \qquad \text{or} \qquad \lim_{x\to 0^-} \sqrt{x}.$$

But it does make sense to try to find

$$\lim_{x\to 0^+} \sqrt{x}. \qquad \text{(Figure 2.2.16)}$$

As you probably suspect, this one-sided limit exists and is 0.

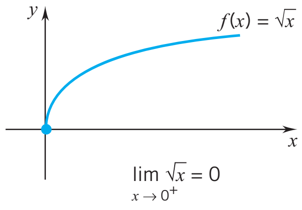


Figure 2.2.16

EXERCISES 2.2

Exercises 1–20. Decide in the manner of Section 2.1 whether or not the indicated limit exists. Evaluate the limits that do exist.

1. $\lim_{x\to 1} \dfrac{x}{x+1}$.

2. $\lim_{x\to 0} \dfrac{x^2(1+x)}{2x}$.

3. $\lim_{x\to 0} \dfrac{x(1+x)}{2x^2}$.

4. $\lim_{x\to 4} \dfrac{x}{\sqrt{x}+1}$.

5. $\lim_{x\to 1} \dfrac{x^4-1}{x-1}$.

6. $\lim_{x\to -1} \dfrac{1-x}{x+1}$.

7. $\lim_{x\to 0} \dfrac{x}{|x|}$.

8. $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$.

9. $\lim_{x\to -2} \dfrac{|x|}{x}$.

10. $\lim_{x\to 9} \dfrac{x-3}{\sqrt{x}-3}$.

11. $\lim_{x\to 3^+} \dfrac{x+3}{x^2-7x+12}$.

12. $\lim_{x\to 0^-} \dfrac{x}{|x|}$.

13. $\lim_{x\to 1^+} \dfrac{\sqrt{x-1}}{x}$.

14. $\lim_{x\to 3^-} \sqrt{9-x^2}$.

15. $\lim_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} 2x-1, & x \le 2 \\ x^2-x, & x > 2. \end{cases}$

16. $\lim_{x\to -1^-} f(x)$ if $f(x) = \begin{cases} 1, & x \le -1 \\ x+2, & x > -1. \end{cases}$

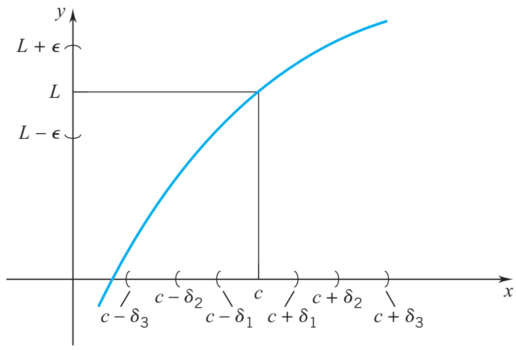
17. $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 3, & x \text{ an integer} \\ 1, & \text{otherwise.} \end{cases}$

18. $\lim_{x\to 3} f(x)$ if $f(x) = \begin{cases} x^2, & x < 3 \\ 7, & x = 3 \\ 2x+3, & x > 3. \end{cases}$

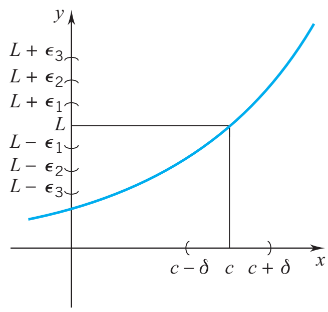
19. $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 3, & x \text{ an integer} \\ 1, & \text{otherwise.} \end{cases}$

20. $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} x^2, & x \le 1 \\ 5x, & x > 1. \end{cases}$

21. Which of the δ's displayed in the figure "works" for the given ϵ?



22. For which of the ϵ's given in the figure does the specified δ work?



Exercises 23–26. Find the largest δ that "works" for the given ϵ.

23. $\lim_{x\to 1} 2x = 2$; $\epsilon = 0.1$.

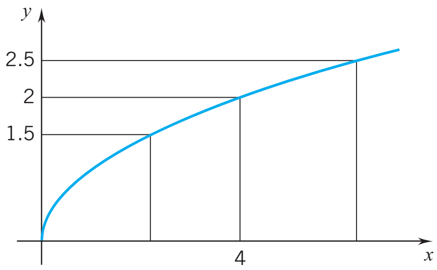
24. $\lim_{x\to 4} 5x = 20$; $\epsilon = 0.5$.

25. $\lim_{x\to 2} \frac{1}{2}x = 1$; $\epsilon = 0.01$.

26. $\lim_{x\to 2} \frac{1}{5}x = \frac{2}{5}$; $\epsilon = 0.1$.

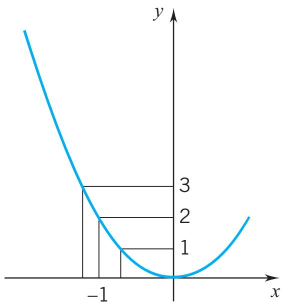
27. The graphs of $f(x) = \sqrt{x}$ and the horizontal lines $y = 1.5$ and $y = 2.5$ are shown in the figure. Use a graphing utility to find a $\delta > 0$ which is such that

$$\text{if} \quad 0 < |x - 4| < \delta, \quad \text{then} \quad |\sqrt{x} - 2| < 0.5.$$



28. The graphs of $f(x) = 2x^2$ and the horizontal lines $y = 1$ and $y = 3$ are shown in the figure. Use a graphing utility to find a $\delta > 0$ which is such that

$$\text{if} \quad 0 < |x + 1| < \delta, \quad \text{then} \quad |2x^2 - 2| < 1.$$



Exercises 29–34. For each of the limits stated and the ϵ's given, use a graphing utility to find a $\delta > 0$ which is such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Draw the graph of f together with the vertical lines $x = c - \delta$, $x = c + \delta$ and the horizontal lines $y = L - \epsilon$, $y = L + \epsilon$.

29. $\lim_{x\to 2} \left(\frac{1}{4}x^2 + x + 1\right) = 4;\ \epsilon = 0.5,\ \epsilon = 0.25.$

30. $\lim_{x\to -2} (x^3 + 4x + 2) = 2;\ \epsilon = 0.5,\ \epsilon = 0.25.$

31. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1} = 2;\ \epsilon = 0.5,\ \epsilon = 0.25.$

32. $\lim_{x\to -1} \dfrac{1 - 3x}{2x + 4} = 2;\ \epsilon = 0.5,\ \epsilon = 0.1.$

33. $\lim_{x\to 0} \dfrac{\sin 3x}{x} = 3;\ \epsilon = 0.25,\ \epsilon = 0.1.$

34. $\lim_{x\to 1} \tan(\pi x/4) = 1; \epsilon = 0.5,\ \epsilon = 0.1.$

Give an ϵ, δ proof for the following statements.

35. $\lim_{x\to 4} (2x - 5) = 3.$

36. $\lim_{x\to 2} (3x - 1) = 5.$

37. $\lim_{x\to 3} (6x - 7) = 11.$

38. $\lim_{x\to 0} (2 - 5x) = 2.$

39. $\lim_{x\to 2} |1 - 3x| = 5.$

40. $\lim_{x\to 2} |x - 2| = 0.$

41. Let f be some function for which you know only that

$$\text{if} \quad 0 < |x - 3| < 1, \quad \text{then} \quad |f(x) - 5| < 0.1.$$

Which of the following statements are necessarily true?

(a) If $|x - 3| < 1$, then $|f(x) - 5| < 0.1$.
(b) If $|x - 2.5| < 0.3$, then $|f(x) - 5| < 0.1$.
(c) $\lim_{x\to 3} f(x) = 5$.
(d) If $0 < |x - 3| < 2$, then $|f(x) - 5| < 0.1$.
(e) If $0 < |x - 3| < 0.5$, then $|f(x) - 5| < 0.1$.
(f) If $0 < |x - 3| < \frac{1}{4}$, then $|f(x) - 5| < \frac{1}{4}(0.1)$.
(g) If $0 < |x - 3| < 1$, then $|f(x) - 5| < 0.2$.
(h) If $0 < |x - 3| < 1$, then $|f(x) - 4.95| < 0.05$.
(i) If $\lim_{x\to 3} f(x) = L$, then $4.9 \le L \le 5.1$.

42. Suppose that $|A - B| < \epsilon$ for each $\epsilon > 0$. Prove that $A = B$. HINT: Suppose that $A \ne B$ and set $\epsilon = \frac{1}{2}|A - B|$.

Exercises 43–44. Give the four limit statements displayed in (2.2.6), taking

43. $f(x) = \dfrac{1}{x - 1},\ c = 3$

44. $f(x) = \dfrac{x}{x^2 + 2},\ c = 1.$

45. Prove that

(2.2.10) $$\lim_{x\to c} f(x) = 0, \quad \text{iff} \quad \lim_{x\to c} |f(x)| = 0.$$

46. (a) Prove that

$$\text{if} \quad \lim_{x\to c} f(x) = L, \quad \text{then} \quad \lim_{x\to c} |f(x)| = |L|.$$

(b) Show that the converse is false. Give an example where

$$\lim_{x\to c} |f(x)| = |L| \quad \text{and} \quad \lim_{x\to c} f(x) = M \ne L,$$

and then give an example where

$$\lim_{x\to c} |f(x)| \quad \text{exists but} \quad \lim_{x\to c} f(x) \quad \text{does not exist.}$$

47. Give an ϵ, δ proof that statement (i) in (2.2.6) is equivalent to (ii).

48. Give an ϵ, δ proof of (2.2.9).

49. (a) Show that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ for each $c > 0$.

HINT: If x and c are positive, then

$$0 \le |\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} < \frac{1}{\sqrt{c}}|x - c|.$$

(b) Show that $\lim_{x\to 0^+} \sqrt{x} = 0.$

Give an ϵ, δ proof for the following statements.

50. $\lim_{x\to 2} x^2 = 4.$

51. $\lim_{x\to 1} x^3 = 1.$

52. $\lim_{x\to 3} \sqrt{x + 1} = 2.$

53. $\lim_{x\to 3^-} \sqrt{3 - x} = 0.$

54. Prove that, for the function

$$g(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}, \end{cases}$$

$\lim_{x\to 0} g(x) = 0.$

55. The function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

is called the *Dirichlet function*. Prove that for no number c does $\lim_{x \to c} f(x)$ exist.

Prove the limit statement.

56. $\lim_{x \to c^-} f(x) = L$ iff $\lim_{h \to 0} f(c - |h|) = L$.

57. $\lim_{x \to c^+} f(x) = L$ iff $\lim_{h \to 0} f(c + |h|) = L$.

58. $\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c} [f(x) - L] = 0$.

59. Suppose that $\lim_{x \to c} f(x) = L$.

(a) Prove that if $L > 0$, then $f(x) > 0$ for all $x \neq c$ in an interval of the form $(c - \gamma, c + \gamma)$.
HINT: Use an ϵ, δ argument, setting $\epsilon = L$.

(b) Prove that if $L < 0$, then $f(x) < 0$ for all $x \neq c$ in an interval of the form $(c - \gamma, c + \gamma)$.

60. Prove or give a counterexample: if $f(c) > 0$ and $\lim_{x \to c} f(x)$ exists, then $f(x) > 0$ for all x in an interval of the form $(c - \gamma, c + \gamma)$.

61. Suppose that $f(x) \leq g(x)$ for all $x \in (c - p, c + p)$, except possibly at c itself.

(a) Prove that $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$, provided each of these limits exist.

(b) Suppose that $f(x) < g(x)$ for all $x \in (c - p, c + p)$, except possibly at c itself. Does it follow that $\lim_{x \to c} f(x) < \lim_{x \to c} g(x)$?

62. Prove that if $\lim_{x \to c} f(x) = L$, then there are positive numbers δ and B such that if $0 < |x - c| < \delta$, then $|f(x)| < B$.

■ 2.3 SOME LIMIT THEOREMS

As you probably gathered by working through the previous section, it can become rather tedious to apply the ϵ, δ definition of limit time and time again. By proving some general theorems, we can avoid some of this repetitive work. Of course, the theorems themselves (at least the first ones) will have to be proved by ϵ, δ methods.

We begin by showing that if a limit exists, it is unique.

THEOREM 2.3.1 THE UNIQUENESS OF A LIMIT

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$, then $L = M$.

PROOF We show $L = M$ by proving that the assumption $L \neq M$ leads to the false conclusion that

$$|L - M| < |L - M|.$$

Assume that $L \neq M$. Then $|L - M|/2 > 0$. Since $\lim_{x \to c} f(x) = L$, we know that there exists a $\delta_1 > 0$ such that

(1) if $0 < |x - c| < \delta_1$, then $|f(x) - L| < |L - M|/2$.

(Here we are using $|L - M|/2$ as ϵ.)

Since $\lim_{x \to c} f(x) = M$, we know that there exists a $\delta_2 > 0$ such that

(2) if $0 < |x - c| < \delta_2$, then $|f(x) - L| < |L - M|/2$.

(Again, we are using $|L - M|/2$ as ϵ.)

Now let x_1 be a number that satisfies the inequality

$$0 < |x_1 - c| < \text{minimum of } \delta_1 \text{ and } \delta_2.$$

Then, by (1) and (2),

$$|f(x_1) - L| < \frac{|L - M|}{2} \quad \text{and} \quad |f(x_1) - M| < \frac{|L - M|}{2}.$$

It follows that

$$\begin{aligned} |L-M| &= |[L-f(x_1)]+[f(x_1)-M]| \\ &\leq |L-f(x_1)|+|f(x_1)-M| \\ &= |f(x_1)-L|+|f(x_1)-M| < \frac{|L-M|}{2}+\frac{|L-M|}{2} = |L-M|. \end{aligned}$$

by the triangle inequality (for the second line); $|a| = |-a|$ (for the third line) ❑

THEOREM 2.3.2

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(i) $\lim_{x\to c} [f(x)+g(x)] = L+M$,

(ii) $\lim_{x\to c} [\alpha f(x)] = \alpha L$ $|\alpha|$ a real number

(iii) $\lim_{x\to c} [f(x)g(x)] = LM$.

PROOF Let $\epsilon > 0$. To prove (i), we must show that there exists a $\delta > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta, \quad \text{then} \quad |[f(x)+g(x)]-[L+M]| < \epsilon.$$

Note that

$$(*) \qquad \begin{aligned} |[f(x)+g(x)]-[L+M]| &= |[f(x)-L]+[g(x)-M]| \\ &\leq |f(x)-L|+|g(x)-M|. \end{aligned}$$

We can make $|[f(x)+g(x)]-[L+M]|$ less than ϵ by making $|f(x)-L|$ and $|g(x)-M|$ each less than $\frac{1}{2}\epsilon$. Since $\epsilon > 0$, we know that $\frac{1}{2}\epsilon > 0$. Since

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M,$$

we know that there exist positive numbers δ_1 and δ_2 such that

$$\text{if} \quad 0 < |x-c| < \delta_1, \quad \text{then} \quad |f(x)-L| < \tfrac{1}{2}\epsilon$$

and

$$\text{if} \quad 0 < |x-c| < \delta_2, \quad \text{then} \quad |g(x)-M| < \tfrac{1}{2}\epsilon.$$

Now we set $\delta =$ the minimum of δ_1 and δ_2 and note that, if $0 < |x-c| < \delta$, then

$$|f(x)-L| < \tfrac{1}{2}\epsilon \quad \text{and} \quad |g(x)-M| < \tfrac{1}{2}\epsilon.$$

Thus, by $(*)$,

$$|[f(x)+g(x)]-[L+M]| < \epsilon.$$

In summary, by setting $\delta = \min\{\delta_1, \delta_2\}$, we find that

$$\text{if} \quad 0 < |x-c| < \delta \quad \text{then} \quad |[f(x)+g(x)]-[L+M]| < \epsilon.$$

This completes the proof of (i). For proofs of (ii) and (iii), see the supplement to this section. ❑

If you are wondering about $\lim_{x\to c}[f(x) - g(x)]$, note that

$$f(x) - g(x) = f(x) + (-1)g(x),$$

and so the result

(2.3.3) $$\lim_{x\to c}[f(x) - g(x)] = L - M$$

follows from (i) and (ii).

Theorem 2.3.2 can be extended (by mathematical induction) to any finite collection of functions; in particular, if

$$\lim_{x\to c} f_1(x) = L_1, \qquad \lim_{x\to c} f_2(x) = L_2, \qquad \ldots, \qquad \lim_{x\to c} f_n(x) = L_n,$$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers, then

(2.3.4) $$\lim_{x\to c}[\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x)] = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_n L_n.$$

Also,

(2.3.5) $$\lim_{x\to c}[f_1(x) f_2(x) \cdots f_n(x)] = L_1 L_2 \cdots L_n.$$

For each polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and each real number c

(2.3.6) $$\lim_{x\to c} P(x) = P(c).$$

PROOF We already know that

$$\lim_{x\to c} x = c.$$

From (2.3.5) we know that

$$\lim_{x\to c} x^k = c^k \qquad \text{for each positive integer } k.$$

We also know that $\lim_{x\to c} a_0 = a_0$. It follows from (2.3.4) that

$$\lim_{x\to c} [a_n x^n + \cdots + a_1 x + a_0] = a_n c^n + \cdots + a_1 c + a_0,$$

which says that

$$\lim_{x\to c} P(x) = P(c).$$

A function f for which $\lim_{x\to c} f(x) = f(c)$ is said to be *continuous* at c. What we just showed is that polynomials are continuous at each number c. Continuous functions, our focus in Section 2.4, have a regularity and a predictability not shared by other functions.

Examples

$$\lim_{x\to 1}(5x^2 - 12x + 2) = 5(1)^2 - 12(1) + 2 = -5,$$

$$\lim_{x\to 0}(14x^5 - 7x^2 + 2x + 8) = 14(0)^5 - 7(0)^2 + 2(0) + 8 = 8,$$

$$\lim_{x\to -1}(2x^3 + x^2 - 2x - 3) = 2(-1)^3 + (-1)^2 - 2(-1) - 3 = -2. \quad \square$$

We come now to reciprocals and quotients.

THEOREM 2.3.7

If $\lim_{x\to c} g(x) = M$ with $M \neq 0$, then $\lim_{x\to c} \dfrac{1}{g(x)} = \dfrac{1}{M}$.

PROOF Given in the supplement to this section. ☐

Examples

$$\lim_{x\to 4}\frac{1}{x^2} = \frac{1}{16}, \qquad \lim_{x\to 2}\frac{1}{x^3 - 1} = \frac{1}{7}, \qquad \lim_{x\to -3}\frac{1}{|x|} = \frac{1}{|-3|} = \frac{1}{3}. \quad \square$$

Once you know that reciprocals present no trouble, quotients become easy to handle.

THEOREM 2.3.8

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ with $M \neq 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

PROOF The key here is to observe that the quotient can be written as a product:

$$\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}.$$

With $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} \dfrac{1}{g(x)} = \dfrac{1}{M}$,

the product rule [part (iii) of Theorem 2.3.2] gives

$$\lim_{x\to c}\frac{f(x)}{g(x)} = L\frac{1}{M} = \frac{L}{M}. \quad \square$$

This theorem on quotients applied to the quotient of two polynomials gives us the limit of a rational function. If $R = P/Q$ where P and Q are polynomials and c is a real number, then

(2.3.9)
$$\lim_{x\to c} R(x) = \lim_{x\to c}\frac{P(x)}{Q(X)} = \frac{P(c)}{Q(c)} = R(c), \qquad \text{provided } Q(c) \neq 0.$$

This says that a rational function is *continuous* at all numbers c where the denominator is different from zero.

Examples

$$\lim_{x\to 2}\frac{3x-5}{x^2+1}=\frac{6-5}{4+1}=\frac{1}{5},\qquad \lim_{x\to 3}\frac{x^3-3x^2}{1-x^2}=\frac{27-27}{1-9}=0.\quad\square$$

There is no point looking for a limit that does not exist. The next theorem gives a condition under which a quotient does not have a limit.

THEOREM 2.3.10

If $\lim_{x\to c} f(x)=L$ with $L\neq 0$ and $\lim_{x\to c} g(x)=0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist.

PROOF Suppose, on the contrary, that there exists a real number K such that

$$\lim_{x\to c}\frac{f(x)}{g(x)}=K.$$

Then

$$L=\lim_{x\to c} f(x)=\lim_{x\to c}\left[g(x)\cdot\frac{f(x)}{g(x)}\right]=\lim_{x\to c} g(x)\cdot\lim_{x\to c}\frac{f(x)}{g(x)}=0\cdot K=0.$$

This contradicts our assumption that $L\neq 0$. $\square$

Examples

From Theorem 2.3.10 you can see that

$$\lim_{x\to 1}\frac{x^2}{x-1},\qquad \lim_{x\to 2}\frac{3x-7}{x^2-4},\qquad\text{and}\qquad \lim_{x\to 0}\frac{5}{x}$$

all fail to exist. $\square$

Now we come to quotients where both the numerator and denominator tend to zero. Such quotients will be particularly important to us as we go on.

Example 1

Evaluate the limits that exist:

(a) $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$, **(b)** $\lim_{x\to 4}\frac{(x^2-3x-4)^2}{x-4}$, **(c)** $\lim_{x\to -1}\frac{x+1}{(2x^2+7x+5)^2}$.

SOLUTION

(a) First we factor the numerator:

$$\frac{x^2-x-6}{x-3}=\frac{(x+2)(x-3)}{x-3}.$$

For $x\neq 3$,

$$\frac{x^2-x-6}{x-3}=x+2.$$

Therefore

$$\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}(x+2)=5.$$

(b) Note that

$$\frac{(x^2-3x-4)^2}{x-4}=\frac{[(x+1)(x-4)]^2}{x-4}=\frac{(x+1)^2(x-4)^2}{x-4}.$$

Thus for $x\neq 4$,

$$\frac{(x^2-3x-4)^2}{x-4}=(x+1)^2(x-4).$$

It follows that

$$\lim_{x\to 4}\frac{(x^2-3x-4)^2}{x-4}=\lim_{x\to 4}(x+1)^2(x-4)=0.$$

(c) Since

$$\frac{x+1}{(2x^2+7x+5)^2}=\frac{x+1}{[(2x+5)(x+1)]^2}=\frac{x+1}{(2x+5)^2(x+1)^2},$$

for $x\neq -1$,

$$\frac{x+1}{(2x^2+7x+5)^2}=\frac{1}{(2x+5)^2(x+1)}.$$

As $x\to -1$, the denominator tends to 0 but the numerator tends to 1. It follows from Theorem 2.3.10 that

$$\lim_{x\to -1}\frac{1}{(2x+5)^2(x+1)}\qquad\text{does not exist.}$$

Therefore

$$\lim_{x\to -1}\frac{x+1}{(2x^2+7x+5)^2}\qquad\text{does not exist.}$$ ❑

Example 2 Justify the following assertions.

(a) $\displaystyle\lim_{x\to 2}\frac{1/x-1/2}{x-2}=-\frac{1}{4}$ **(b)** $\displaystyle\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}=6.$

SOLUTION

(a) For $x\neq 2$,

$$\frac{1/x-1/2}{x-2}=\frac{\dfrac{2-x}{2x}}{x-2}=\frac{-(x-2)}{2x(x-2)}=\frac{-1}{2x}.$$

Thus

$$\lim_{x\to 2}\frac{1/x-1/2}{x-2}=\lim_{x\to 2}\left[\frac{-1}{2x}\right]=-\frac{1}{4}.$$

(b) Before working with the fraction, we remind you that for each positive number c

$$\lim_{x\to c}\sqrt{x}=\sqrt{c}.\qquad\text{(Exercise 49, Section 2.2)}$$

Now to the fraction. First we "rationalize" the denominator:

$$\frac{x-9}{\sqrt{x}-3} = \frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{(x-9)(\sqrt{x}+3)}{x-9} = \sqrt{x}+3 \quad (x \neq 9).$$

It follows that

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9}[\sqrt{x}+3] = 6. \quad \square$$

Remark In this section we phrased everything in terms of two-sided limits. Although we won't stop to prove it, *analogous results carry over to one-sided limits.* ❑

EXERCISES 2.3

1. Given that

$$\lim_{x\to c} f(x) = 2, \qquad \lim_{x\to c} g(x) = -1, \qquad \lim_{x\to c} h(x) = 0,$$

evaluate the limits that exist. If the limit does not exist, state how you know that.

(a) $\lim_{x\to c}[f(x)-g(x)]$. (b) $\lim_{x\to c}[f(x)]^2$.

(c) $\lim_{x\to c}\frac{f(x)}{g(x)}$. (d) $\lim_{x\to c}\frac{h(x)}{f(x)}$.

(e) $\lim_{x\to c}\frac{f(x)}{h(x)}$. (f) $\lim_{x\to c}\frac{1}{f(x)-g(x)}$.

2. Given that

$$\lim_{x\to c} f(x) = 3, \qquad \lim_{x\to c} g(x) = 0, \qquad \lim_{x\to c} h(x) = -2,$$

evaluate the limits that exist. If the limit does not exist, state how you know that.

(a) $\lim_{x\to c}[3f(x)-2h(x)]$. (b) $\lim_{x\to c}[h(x)]^3$.

(c) $\lim_{x\to c}\frac{h(x)}{x-c}$. (d) $\lim_{x\to c}\frac{g(x)}{h(x)}$.

(e) $\lim_{x\to c}\frac{4}{f(x)-h(x)}$. (f) $\lim_{x\to c}[3+g(x)]^2$.

3. When asked to evaluate

$$\lim_{x\to 4}\left(\frac{1}{x}-\frac{1}{4}\right)\left(\frac{1}{x-4}\right),$$

Moe replies that the limit is zero since $\lim_{x\to 4}\left[\frac{1}{x}-\frac{1}{4}\right] = 0$ and cites Theorem 2.3.2 as justification. Verify that the limit is actually $-\frac{1}{16}$ and identify Moe's error.

4. When asked to evaluate

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3},$$

Moe says that the limit does not exist since $\lim_{x\to 3}(x-3) = 0$ and cites Theorem 2.3.10 (limit of a quotient) as justification. Verify that the limit is actually 7 and identify Moe's error.

Exercises 5–38. Evaluate the limits that exist.

5. $\lim_{x\to 2} 3$.

6. $\lim_{x\to 3}(5-4x)^2$.

7. $\lim_{x\to -4}(x^2+3x-7)$.

8. $\lim_{x\to -2} 3|x-1|$.

9. $\lim_{x\to\sqrt{3}}|x^2-8|$.

10. $\lim_{x\to -1}\frac{x^2+1}{3x^5+4}$.

11. $\lim_{x\to 0}\left(x-\frac{4}{x}\right)$.

12. $\lim_{x\to 5}\frac{2-x^2}{4x}$.

13. $\lim_{x\to 0}\frac{x^2+1}{x-1}$.

14. $\lim_{x\to 0}\frac{x^2}{x^2+1}$.

15. $\lim_{x\to 2}\frac{x}{x^2-4}$.

16. $\lim_{h\to 0} h\left(1-\frac{1}{h}\right)$.

17. $\lim_{h\to 0} h\left(1+\frac{1}{h}\right)$.

18. $\lim_{x\to 2}\frac{x-2}{x^2-4}$.

19. $\lim_{x\to 2}\frac{x^2-4}{x-2}$.

20. $\lim_{x\to -2}\frac{(x^2-x-6)^2}{x+2}$.

21. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$.

22. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$.

23. $\lim_{x\to 1}\frac{x^2-x-6}{(x+2)^2}$.

24. $\lim_{x\to -2}\frac{x^2-x-6}{(x+2)^2}$.

25. $\lim_{h\to 0}\frac{1-1/h^2}{1-1/h}$.

26. $\lim_{h\to 0}\frac{1-1/h^2}{1+1/h^2}$.

27. $\lim_{h\to 0}\frac{1-1/h}{1+1/h}$.

28. $\lim_{h\to 0}\frac{1+1/h}{1+1/h^2}$.

29. $\lim_{t\to -1}\frac{t^2+6t+5}{t^2+3t+2}$.

30. $\lim_{x\to 2^+}\frac{\sqrt{x^2-4}}{x-2}$.

31. $\lim_{t\to 0}\frac{t+a/t}{t+b/t}$.

32. $\lim_{x\to 1}\frac{x^2-1}{x^3-1}$.

33. $\lim_{x\to 1}\frac{x^5-1}{x^4-1}$.

34. $\lim_{h\to 0} h^2\left(1+\frac{1}{h}\right)$.

35. $\lim_{h\to 0} h\left(1+\frac{1}{h^2}\right)$.

36. $\lim_{x\to -4}\left(\frac{3x}{x+4}+\frac{8}{x+4}\right)$.

37. $\lim_{x\to -4}\left(\frac{2x}{x+4}+\frac{8}{x+4}\right)$.

38. $\lim_{x\to -4}\left(\frac{2x}{x+4}-\frac{8}{x+4}\right)$.

39. Evaluate the limits that exist.

(a) $\lim_{x\to 4} \left(\frac{1}{x} - \frac{1}{4}\right)$.

(b) $\lim_{x\to 4} \left[\left(\frac{1}{x} - \frac{1}{4}\right)\left(\frac{1}{x-4}\right)\right]$.

(c) $\lim_{x\to 4} \left[\left(\frac{1}{x} - \frac{1}{4}\right)(x-2)\right]$.

(d) $\lim_{x\to 4} \left[\left(\frac{1}{x} - \frac{1}{4}\right)\left(\frac{1}{x-4}\right)^2\right]$.

40. Evaluate the limits that exist.

(a) $\lim_{x\to 3} \frac{x^2+x+12}{x-3}$. (b) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$.

(c) $\lim_{x\to 3} \frac{(x^2+x-12)^2}{x-3}$. (d) $\lim_{x\to 3} \frac{x^2+x-12}{(x-3)^2}$.

41. Given that $f(x) = x^2 - 4x$, evaluate the limits that exist.

(a) $\lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$. (b) $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$.

(c) $\lim_{x\to 3} \frac{f(x)-f(1)}{x-3}$. (d) $\lim_{x\to 3} \frac{f(x)-f(2)}{x-3}$.

42. Given that $f(x) = x^3$, evaluate the limits that exist.

(a) $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$. (b) $\lim_{x\to 3} \frac{f(x)-f(2)}{x-3}$.

(c) $\lim_{x\to 3} \frac{f(x)-f(3)}{x-2}$. (d) $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$.

43. Show by example that $\lim_{x\to c}[f(x)+g(x)]$ can exist even if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

44. Show by example that $\lim_{x\to c}[f(x)g(x)]$ can exist even if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

Exercises 45–51. True or false? Justify your answers.

45. If $\lim_{x\to c}[f(x)+g(x)]$ exists but $\lim_{x\to c} f(x)$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

46. If $\lim_{x\to c}[f(x)+g(x)]$ and $\lim_{x\to c} f(x)$ exist, then it can happen that $\lim_{x\to c} g(x)$ does not exist.

47. If $\lim_{x\to c} \sqrt{f(x)}$ exists, then $\lim_{x\to c} f(x)$ exists.

48. If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \sqrt{f(x)}$ exists.

49. If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \frac{1}{f(x)}$ exists.

50. If $f(x) \le g(x)$ for all $x \ne c$, then $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$.

51. If $f(x) < g(x)$ for all $x \ne c$, then $\lim_{x\to c} f(x) < \lim_{x\to c} g(x)$.

52. (a) Verify that

$$\max\{f(x), g(x)\} = \tfrac{1}{2}\{[f(x)+g(x)] + |f(x)-g(x)|\}.$$

(b) Find a similar expression for $\min\{f(x), g(x)\}$.

53. Let $h(x) = \min\{f(x), g(x)\}$ and $H(x) = \max\{f(x), g(x)\}$. Show that

$$\text{if} \quad \lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = L,$$

$$\text{then} \quad \lim_{x\to c} h(x) = L \quad \text{and} \quad \lim_{x\to c} H(x) = L.$$

HINT: Use Exercise 52.

54. (*Stability of limit*) Let f be a function defined on some interval $(c-p, c+p)$. Now change the value of f at a finite number of points $x_1, x_2, \ldots, x_n$ and call the resulting function g.

(a) Show that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} g(x) = L$.

(b) Show that if $\lim_{x\to c}$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

55. (a) Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c}[f(x)g(x)] = 1$. Prove that $\lim_{x\to c} g(x)$ does not exist.

(b) Suppose that $\lim_{x\to c} f(x) = L \ne 0$ and $\lim_{x\to c}[f(x)g(x)] = 1$. Does $\lim_{x\to c} g(x)$ exist, and if so, what is it?

56. Let f be a function defined at least on an interval $(c-p, c+p)$. Suppose that for each function g

$$\lim_{x\to c}[f(x)+g(x)] \quad \text{does not exist if} \quad \lim_{x\to c} g(x)$$

does not exist.

Show that $\lim_{x\to c} f(x)$ does exist.

(*Difference quotients*) Let f be a function and let c and $c+h$ be numbers in an interval on which f is defined. The expression

$$\frac{f(c+h)-f(c)}{h}$$

is called a *difference quotient* for f. (Limits of difference quotients as $h \to 0$ are at the core of Chapter 3.) In Exercises 57–60, calculate

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

for the function f and the number c.

57. $f(x) = 2x^2 - 3x$; $c = 2$.

58. $f(x) = x^3 + 1$; $c = -1$.

59. $f(x) = \sqrt{x}$; $c = 4$.

60. $f(x) = 1/(x+1)$; $c = 1$.

61. Calculate

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for each of the following functions:

(a) $f(x) = x$.

(b) $f(x) = x^2$.

(c) $f(x) = x^3$.

(d) $f(x) = x^4$.

(e) $f(x) = x^n$, n an arbitrary positive integer. Make a guess and confirm your guess by induction.

*SUPPLEMENT TO SECTION 2.3

PROOF OF THEOREM 2.3.2 (II)

We consider two cases: $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$, then $\epsilon/|\alpha| > 0$ and, since

$$\lim_{x\to c} f(x) = L,$$

we know that there exists $\delta > 0$ such that,

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \frac{\epsilon}{|\alpha|}.$$

From the last inequality, we obtain

$$|\alpha||f(x) - L| < \epsilon \quad \text{and thus} \quad |\alpha f(x) - \alpha L| < \epsilon.$$

The case $\alpha = 0$ was treated before. (2.2.5) ❑

PROOF OF THEOREM 2.3.2 (III)

We begin with a little algebra:

$$\begin{aligned}
|f(x)g(x) - LM| &= |[f(x)g(x) - f(x)M] + [f(x)M - LM]| \\
&\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\
&= |f(x)||g(x) - M| + |M||f(x) - L| \\
&\leq |f(x)||g(x) - M| + (1 + |M|)|f(x) - L|.
\end{aligned}$$

Now let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, we know the following:

1. There exists $\delta_1 > 0$ such that, if $0 < |x - c| < \delta_1$, then

$$|f(x) - L| < 1 \quad \text{and thus} \quad |f(x)| < 1 + |L|.$$

2. There exists $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta_2, \quad \text{then} \quad |g(x) - M| < \left(\frac{\frac{1}{2}\epsilon}{1 + |L|}\right).$$

3. There exists $\delta_3 > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta_3, \quad \text{then} \quad |f(x) - L| < \left(\frac{\frac{1}{2}\epsilon}{1 + |M|}\right).$$

We now set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and observe that, if $0 < |x - c| < \delta$, then

$$\begin{aligned}
|f(x) - LM| &\leq |f(x)||g(x) - M| + (1 + |M|)|f(x) - L| \\
&< \underbrace{(1 + |L|)}_{\text{by (1)}}\underbrace{\left(\frac{\frac{1}{2}\epsilon}{1 + |L|}\right)}_{\text{by (2)}} + (1 + |M|)\underbrace{\left(\frac{\frac{1}{2}\epsilon}{1 + |M|}\right)}_{\text{by (3)}} = \epsilon.
\end{aligned}$$ ❑

PROOF OF THEOREM 2.3.7

For $g(x) \neq 0$,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|g(x)||M|}.$$

Choose $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta_1, \quad \text{then} \quad |g(x) - M| < \frac{|M|}{2}.$$

For such x,

$$|g(x)| > \frac{|M|}{2} \qquad \text{so that} \qquad \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and thus

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|g(x)||M|} \leq \frac{2}{|M|^2}|g(x) - M| = \frac{2}{M^2}|g(x) - M|.$$

Now let $\epsilon > 0$ and choose $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta_2, \quad \text{then} \quad |g(x) - M| < \frac{M^2}{2}\epsilon.$$

Setting $\delta = \min\{\delta_1, \delta_2\}$, we find that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon. \quad \square$$

2.4 CONTINUITY

In ordinary language, to say that a certain process is "continuous" is to say that it goes on without interruption and without abrupt changes. In mathematics the word "continuous" has much the same meaning.

The concept of continuity is so important in calculus and its applications that we discuss it with some care. First we treat *continuity at a point c* (a number c), and then we discuss *continuity on an interval*.

Continuity at a Point

The basic idea is as follows: We are given a function f and a number c. We calculate (if we can) both $\lim_{x\to c} f(x)$ and $f(c)$. If these two numbers are equal, we say that f is *continuous* at c. Here is the definition formally stated.

DEFINITION 2.4.1

Let f be a function defined at least on an open interval $(c - p, c + p)$. We say that *f is continuous at c* if

$$\lim_{x\to c} f(x) = f(c).$$

If the domain of f contains an interval $(c - p, c + p)$, then f can fail to be continuous at c for only one of two reasons: either

(i) f has a limit as x tends to c, but $\lim_{x\to c} f(x) \neq f(c)$, *or*

(ii) f has no limit as x tends to c.

In case (i) the number c is called a *removable* discontinuity. The discontinuity can be removed by redefining f at c. If the limit is L, redefine f at c to be L.

In case (ii) the number c is called an *essential* discontinuity. You can change the value of f at a billion points in any way you like. The discontinuity will remain. (Exercise 51.)

The function depicted in Figure 2.4.1 has a removable discontinuity at c. The discontinuity can be removed by lowering the dot into place (i.e., by redefining f at c to be L).

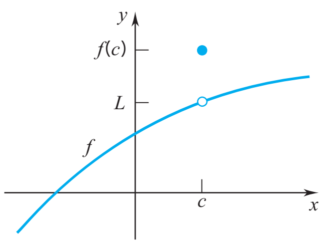


Figure 2.4.1

The functions depicted in Figures 2.4.2, 2.4.3, and 2.4.4 have essential discontinuities at c. The discontinuity in Figure 2.4.2 is, for obvious reasons, called a *jump* discontinuity. The functions of Figure 2.4.3 have *infinite* discontinuities.

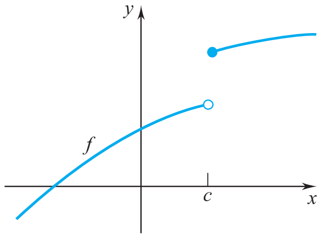


Figure 2.4.2

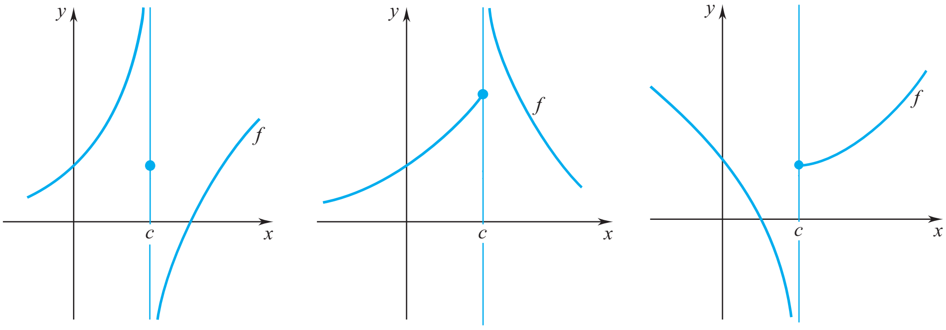


Figure 2.4.3

In Figure 2.4.4, we have tried to portray the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} \end{cases}$$

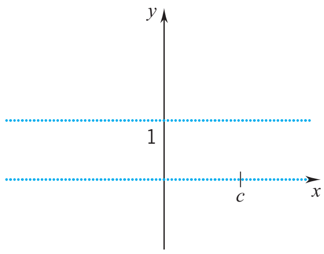


Figure 2.4.4

At no point c does f have a limit. Each point is an essential discontinuity. The function is everywhere discontinuous.

Most of the functions that you have encountered so far are continuous at each point of their domains. In particular, this is true for polynomials P,

$$\lim_{x \to c} P(x) = P(c), \qquad [(2.3.6)]$$

for rational functions (quotients of polynomials) $R = P/Q$,

$$\lim_{x \to c} R(x) = \lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = R(c) \quad \text{provided} \quad Q(c) \neq 0, \qquad [(2.3.9)]$$

and for the absolute value function,

$$\lim_{x \to c} |x| = |c|. \qquad [(2.2.4)]$$

As you were asked to show earlier (Exercise 49, Section 2.2),

$$\lim_{x \to c} \sqrt{x} = \sqrt{c} \qquad \text{for each } c > 0.$$

This makes the square-root function continuous at each positive number. What happens at $c = 0$, we discuss later.

With f and g continuous at c, we have

$$\lim_{x \to c} f(x) = f(c) \qquad \lim_{x \to c} g(x) = g(c)$$

and thus, by the limit theorems,

$$\lim_{x \to c} [f(x) + g(x)] = f(c) + g(c), \qquad \lim_{x \to c} [f(x) - g(x)] = f(c) - g(c)$$

$$\lim_{x \to c} [\alpha f(x)] = \alpha f(c) \quad \text{for each real } \alpha \qquad \lim_{x \to c} [f(x)g(x)] = f(c)g(c)$$

$$\text{and, if } g(c) \neq 0, \qquad \lim_{x \to c} [f(x)/g(x)] = f(c)/g(c).$$

We summarize all this in a theorem.

THEOREM 2.4.2

If f and g are continuous at c, then

(i) $f + g$ is continuous at c;
(ii) $f - g$ is continuous at c;
(iii) αf is continuous at c for each real α;
(iv) $f \cdot g$ is continuous at c;
(v) f/g is continuous at c provided $g(c) \neq 0$.

These results can be combined and extended to any finite number of functions.

Example 1 The function $F(x) = 3|x| + \dfrac{x^3 - x}{x^2 - 5x + 6} + 4$ is continuous at all real numbers other than 2 and 3. You can see this by noting that

$$F = 3f + g/h + k$$

where

$$f(x) = |x|, \qquad g(x) = x^3 - x, \qquad h(x) = x^2 - 5x + 6, \qquad k(x) = 4.$$

Since f, g, h, k are everywhere continuous, F is continuous except at 2 and 3, the numbers at which h takes on the value 0. (At those numbers F is not defined.) ❑

Our next topic is the continuity of composite functions. Before getting into this, however, let's take a look at continuity in terms of ϵ, δ. A direct translation of

$$\lim_{x \to c} f(x) = f(c)$$

into ϵ, δ terms reads like this: for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - f(c)| < \epsilon.$$

Here the restriction $0 < |x - c|$ is unnecessary. We can allow $|x - c| = 0$ because then $x = c$, $f(x) = f(c)$, and thus $|f(x) - f(c)| = 0$. Being 0, $|f(x) - f(c)|$ is certainly less than ϵ.

Thus, an ϵ, δ characterization of continuity at c reads as follows:

(2.4.3) $$f \text{ is continuous at } c \text{ if } \begin{cases} \text{for each } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ such that} \\ \text{if} \quad |x - c| < \delta, \quad \text{then} \quad |f(x) - f(c)| < \epsilon. \end{cases}$$

In intuitive terms

$$f \text{ is continuous at } c \qquad \text{if} \qquad \text{for } x \text{ close to } c, \qquad f(x) \text{ is close to } f(c).$$

We are now ready to take up the continuity of composite functions. Remember the defining formula: $(f \circ g)(x) = f(g(x))$. (You may wish to review Section 1.7.)

THEOREM 2.4.4

If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

The idea here is as follows: with g continuous at c, we know that

for x close to c, $g(x)$ is close to $g(c)$;

from the continuity of f at $g(c)$, we know that

with $g(x)$ close to $g(c)$, $f(g(x))$ is close to $f(g(c))$.

In summary,

with x close to c, $f(g(x))$ is close to $f(g(c))$.

The argument we just gave is too vague to be a proof. Here, in contrast, is a proof. We begin with $\epsilon > 0$. We must show that there exists a number $\delta > 0$ such that

$$\text{if} \quad |x - c| < \delta, \quad \text{then} \quad |f(g(x)) - f(g(c))| < \epsilon.$$

In the first place, we observe that, since f is continuous at $g(c)$, there does exist a number $\delta_1 > 0$ such that

$$(1) \qquad \text{if} \quad |t - g(c)| < \delta_1, \quad \text{then} \quad |f(t) - f(g(c))| < \epsilon.$$

With $\delta_1 > 0$, we know from the continuity of g at c that there exists a number $\delta > 0$ such that

$$(2) \qquad \text{if} \quad |x - c| < \delta, \quad \text{then} \quad |g(x) - g(c)| < \delta_1.$$

Combining (2) and (1), we have what we want: by (2),

$$\text{if} \quad |x - c| < \delta, \quad \text{then} \quad |g(x) - g(c)| < \delta_1$$

so that by (1)

$$|f(g(x)) - f(g(c))| < \epsilon.$$

This proof is illustrated in Figure 2.4.5. The numbers within δ of c are taken by g to within δ_1 of $g(c)$, and then by f to within ϵ of $f(g(c))$.

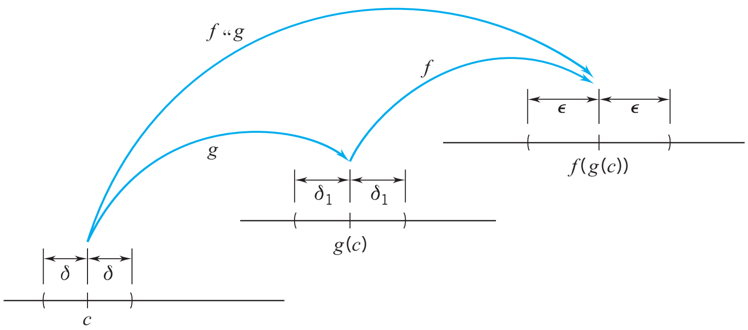


Figure 2.4.5

It's time to look at some examples.

Example 2 The function $F(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous at all numbers greater than 3. To see this, note that $F = f \circ g$, where

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \frac{x^2+1}{x-3}.$$

Now, take any $c > 3$. Since g is a rational function and g is defined at c, g is continuous at c. Also, since $g(c)$ is positive and f is continuous at each positive number, f is continuous at $g(c)$. By Theorem 2.4.4, F is continuous at c. ❑

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous *where it is applied.*

Example 3 The function $F(x) = \dfrac{1}{5 - \sqrt{x^2+16}}$ is continuous everywhere except at $x = \pm 3$, where it is not defined. To see this, note that $F = f \circ g \circ k \circ h$, where

$$f(x) = \frac{1}{x}, \qquad g(x) = 5 - x, \qquad k(x) = \sqrt{x}, \qquad h(x) = x^2 + 16,$$

and observe that each of these functions is being evaluated only where it is continuous. In particular, g and h are continuous everywhere, f is being evaluated only at nonzero numbers, and k is being evaluated only at positive numbers. ❑

Just as we considered one-sided limits, we can consider one-sided continuity.

DEFINITION 2.4.5 ONE-SIDED CONTINUITY

A function f is called

continuous from the left at c if $\lim_{x \to c^-} f(x) = f(c)$.

It is called

continuous from the right at c if $\lim_{x \to c^+} f(x) = f(c)$.

The function of Figure 2.4.6 is continuous from the right at 0; the function of Figure 2.4.7 is continuous from the left at 1.

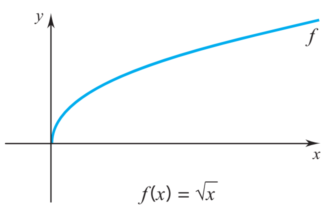


Figure 2.4.6

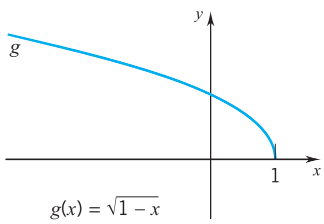


Figure 2.4.7

It follows from (2.2.9) that a function is continuous at c iff it is continuous from both sides at c. Thus

(2.4.6) f is continuous at c iff $f(c)$, $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ all exist and are equal.

Example 4 Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 1, & 0 < x \le 1 \\ x^2+1, & x > 1. \end{cases} \qquad \text{(Figure 2.4.8)}$$

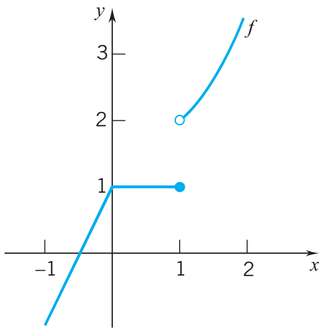


Figure 2.4.8

SOLUTION Clearly f is continuous at each point in the open intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$. (On each of these intervals f is a polynomial.) Thus, we have to check the behavior of f at $x = 0$ and $x = 1$. The figure suggests that f is continuous at 0 and discontinuous at 1. Indeed, that is the case:

$$f(0) = 1, \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 1, \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1) = 1.$$

This makes f continuous at 0. The situation is different at $x = 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1) = 1 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 2.$$

Thus f has an essential discontinuity at 1, a jump discontinuity. ❑

Example 5 Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} x^3, & x \le -1 \\ x^2-2, & -1 < x < 1 \\ 6-x, & 1 \le x < 4 \\ \dfrac{6}{7-x}, & 4 < x < 7 \\ 5x+2, & x \ge 7. \end{cases}$$

SOLUTION It should be clear that f is continuous at each point of the open intervals $(-\infty, -1)$, $(-1, 1)$, $(1, 4)$, $(4, 7)$, $(7, \infty)$. All we have to check is the behavior of f at $x = -1, 1, 4, 7$. To do so, we apply (2.4.6).

The function is continuous at $x = -1$ since $f(-1) = (-1)^3 = -1$,

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^3) = -1, \quad \text{and} \quad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x^2-2) = -1.$$

Our findings at the other three points are displayed in the following chart. Try to verify each entry.

c	$f(c)$	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	Conclusion
1	5	−1	5	discontinuous
4	not defined	2	2	discontinuous
7	37	does not exist	37	discontinuous

The discontinuity at $x = 4$ is removable: if we redefine f at 4 to be 2, then f becomes continuous at 4. The numbers 1 and 7 are essential discontinuities. The discontinuity at 1 is a jump discontinuity; the discontinuity at 7 is an infinite discontinuity: $f(x) \to \infty$ as $x \to 7^-$. ❑

Continuity on Intervals

A function f is said to be *continuous on an interval* if it is continuous at each interior point of the interval and one-sidedly continuous at whatever endpoints the interval may contain.

For example:

(i) The function

$$f(x) = \sqrt{1 - x^2}$$

is continuous on $[-1, 1]$ because it is continuous at each point of $(-1, 1)$, continuous from the right at -1, and continuous from the left at 1. The graph of the function is the semicircle shown in Figure 2.4.9.

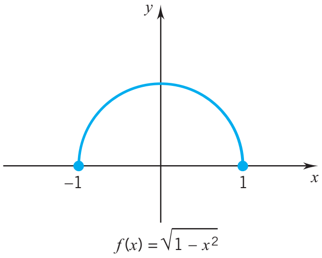


Figure 2.4.9

(ii) The function

$$f(x) = \frac{1}{\sqrt{1 - x^2}}$$

is continuous on $(-1, 1)$ because it is continuous at each point of $(-1, 1)$. It is not continuous on $[-1, 1)$ because it is not continuous from the right at -1. It is not continuous on $(-1, 1]$ because it is not continuous from the left at 1.

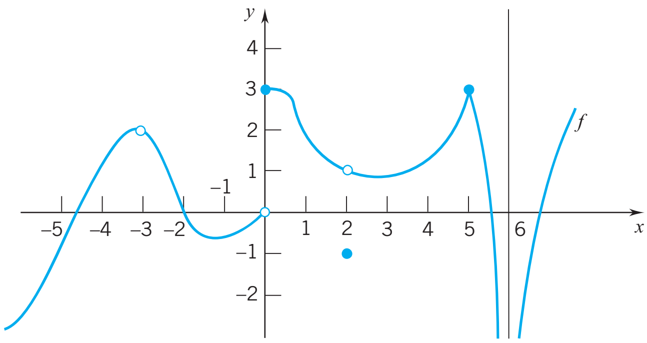
(iii) The function graphed in Figure 2.4.8 is continuous on $(-\infty, 1]$ and continuous on $(1, \infty)$. It is not continuous on $[1, \infty)$ because it is not continuous from the right at 1.

(iv) Polynomials, being everywhere continuous, are continuous on $(-\infty, \infty)$.

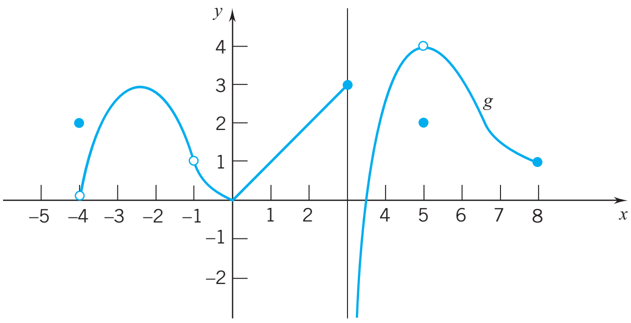
Continuous functions have special properties not shared by other functions. Two of these properties are featured in Section 2.6. Before we get to these properties, we prove a very useful theorem and revisit the trigonometric functions.

EXERCISES 2.4

1. The graph of f is given in the figure.
(a) At which points is f discontinuous?
(b) For each point of discontinuity found in (a), determine whether f is continuous from the right, from the left, or neither.
(c) Which, if any, of the points of discontinuity found in (a) is removable? Which, if any, is a jump discontinuity?



2. The graph of g is given in the figure. Determine the intervals on which g is continuous.



Exercises 3–16. Determine whether or not the function is continuous at the indicated point. If not, determine whether the discontinuity is a removable discontinuity or an essential discontinuity. If the latter, state whether it is a jump discontinuity, an infinite discontinuity, or neither.

3. $f(x) = x^3 - 5x + 1; \quad x = 2.$

4. $g(x) = \sqrt{(x-1)^2 + 5}; \quad x = 1.$

5. $f(x) = \sqrt{x^2 + 9}; \quad x = 3.$

6. $f(x) = |4 - x^2|; \quad x = 2.$

7. $f(x) = \begin{cases} x^2 + 4, & x < 2 \\ x^3, & x \geq 2; \end{cases} \quad x = 2.$

8. $h(x) = \begin{cases} x^2 + 5, & x < 2 \\ x^3, & x \geq 2; \end{cases} \quad x = 2.$

9. $g(x) = \begin{cases} x^2 + 4, & x < 2 \\ 5, & x = 2 \\ x^3, & x > 2; \end{cases} \quad x = 2.$

10. $g(x) = \begin{cases} x^2 + 5, & x < 2 \\ 10, & x = 2 \\ 1 + x^3, & x > 2; \end{cases} \quad x = 2.$

11. $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1 \\ 0, & x = 1; \end{cases} \quad x = 1.$

12. $f(x) = \begin{cases} 1 - x, & x < 1 \\ 1, & x = 1; \\ x^2 - 1, & x > 1; \end{cases} \quad x = 1.$

13. $h(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1}, & x \neq -1, \\ -2, & x = -1; \end{cases} \quad x = -1.$

14. $g(x) = \begin{cases} \dfrac{1}{x + 1}, & x \neq -1 \\ 0, & x = -1; \end{cases} \quad x = -1.$

15. $f(x) = \begin{cases} \dfrac{x + 2}{x^2 - 4}, & x \neq 2 \\ 4, & x = 2; \end{cases} \quad x = 2$

16. $f(x) = \begin{cases} -x^2, & x < 0 \\ 0, & x = 0 \\ 1/x^2, & x > 0; \end{cases} \quad x = 0$

Exercises 17–28. Sketch the graph and classify the discontinuities (if any) as being removable or essential. If the latter, is it a jump discontinuity, an infinite discontinuity, or neither.

17. $f(x) = |x - 1|.$

18. $h(x) = |x^2 - 1|.$

19. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2. \end{cases}$

20. $f(x) = \begin{cases} \dfrac{x - 3}{x^2 - 9}, & x \neq 3, -3 \\ \frac{1}{6}, & x = 3, -3 \end{cases}$

21. $f(x) = \begin{cases} \dfrac{x + 2}{x^2 - x - 6}, & x \neq -2, 3 \\ -\frac{1}{5}, & x = -2, 3. \end{cases}$

22. $g(x) = \begin{cases} 2x - 1, & x < 1 \\ 0, & x = 1 \\ 1/x^2, & x > 1. \end{cases}$

23. $f(x) = \begin{cases} -1, & x < -1 \\ x^3, & -1 \leq x \leq 1 \\ 1, & 1 < x. \end{cases}$

24. $g(x) = \begin{cases} 1, & x \leq -2 \\ \frac{1}{2}x, & -2 < x < 4 \\ \sqrt{x}, & 4 \leq x. \end{cases}$

25. $h(x) = \begin{cases} 1, & x \leq 0 \\ x^2, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \\ x, & 2 \leq x. \end{cases}$

26. $g(x) = \begin{cases} -x^2, & x < -1 \\ 3, & x = -1 \\ 2 - x, & -1 < x \leq 1 \\ 1/x^2, & 1 < x. \end{cases}$

27. $f(x) = \begin{cases} 2x + 9, & x < -2 \\ x^2 + 1, & -2 < x \leq 1, \\ 3x - 1, & 1 < x < 3 \\ x + 6, & 3 < x. \end{cases}$

28. $g(x) = \begin{cases} x + 7, & x < -3 \\ |x - 2|, & -3 < x < -1 \\ x^2 - 2x, & -1 < x < 3 \\ 2x - 3, & 3 \leq x. \end{cases}$

29. Sketch a graph of a function f that satisfies the following conditions:
1. $\text{dom}(f) = [-3, 3]$.
2. $f(-3) = f(-1) = 1; f(2) = f(3) = 2.$
3. f has an infinite discontinuity at -1 and a jump discontinuity at 2.
4. f is right continuous at -1 and left continuous at 2.

30. Sketch a graph of a function f that satisfies the following conditions:
1. $\text{dom}(f) = [-2, 2]$.
2. $f(-2) = f(-1) = f(1) = f(2) = 0.$
3. f has an infinite discontinuity at -2, a jump discontinuity at -1, a jump discontinuity at 1, and an infinite discontinuity at 2.
4. f is continuous from the right at -1 and continuous from the left at 1.

Exercises 31–34. If possible, define the function at 1 so that it becomes continuous at 1.

31. $f(x) = \dfrac{x^2 - 1}{x - 1}.$

32. $f(x) = \dfrac{1}{x - 1}.$

33. $f(x) = \dfrac{x - 1}{|x - 1|}.$

34. $f(x) = \dfrac{(x - 1)^2}{|x - 1|}.$

35. Let $f(x) = \begin{cases} x^2, & x < 1 \\ Ax - 3, & x \geq 1. \end{cases}$ Find A given that f is continuous at 1.

36. Let $f(x) = \begin{cases} A^2x^2, & x \leq 2 \\ (1 - A)x, & x > 2. \end{cases}$ For what values of A is f continuous at 2?

37. Give necessary and sufficient conditions on A and B for the function

$$f(x) = \begin{cases} Ax - B, & x \leq 1 \\ 3x, & 1 < x < 2 \\ Bx^2 - A, & 2 \leq x \end{cases}$$

to be continuous at $x = 1$ but discontinuous at $x = 2$.

38. Give necessary and sufficient conditions on A and B for the function in Exercise 37 to be continuous at $x = 2$ but discontinuous at $x = 1$.

39. Set $f(x) = \begin{cases} 1 + cx, & x < 2 \\ c - x, & x \geq 2. \end{cases}$ Find a value of c that makes f continuous on $(-\infty, \infty)$. Use a graphing utility to verify your result.

40. Set $f(x) = \begin{cases} 1 - cx + dx^2, & x \leq -1 \\ x^2 + x, & -1 < x < 2 \\ cx^2 + dx + 4, & x \geq 2. \end{cases}$ Find values of c and d that make f continuous on $(-\infty, \infty)$. Use a graphing utility to verify your result.

Exercises 41–44. Define the function at 5 so that it becomes continuous at 5.

41. $f(x) = \dfrac{\sqrt{x+4} - 3}{x - 5}$.

42. $f(x) = \dfrac{\sqrt{x+4} - 3}{\sqrt{x} - 5}$.

43. $f(x) = \dfrac{\sqrt{2x-1} - 3}{x - 5}$.

44. $f(x) = \dfrac{\sqrt{x^2 - 7x + 16} - \sqrt{6}}{(x - 5)\sqrt{x+1}}$.

Exercises 45–47. At what points (if any) is the function continuous?

45. $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

46. $g(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

47. $h(x) = \begin{cases} 2x, & x \text{ an integer} \\ x^2, & \text{otherwise.} \end{cases}$

48. The following functions are important in science and engineering:

1. The *Heaviside function* $H_c(x) = \begin{cases} 0, & x < c \\ 1, & x \geq c. \end{cases}$

2. The *unit pulse function*

$$P_{\epsilon,c}(x) = \frac{1}{\epsilon}[H_c(x) - H_{c+\epsilon}(x)].$$

(a) Graph H_c and $P_{\epsilon,c}$.
(b) Determine where each of the functions is continuous.
(c) Find $\lim_{x\to c^-} H_c(x)$ and $\lim_{x\to c^+} H_c(x)$. What can you say about $\lim_{x\to c} H(x)$?

49. (*Important*) Prove that

$$f \text{ is continuous at } c \quad \text{iff} \quad \lim_{h\to 0} f(c+h) = f(c).$$

50. (*Important*) Let f and g be continuous at c. Prove that if:
(a) $f(c) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
(b) $f(c) < 0$, then there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta)$.
(c) $f(c) < g(c)$, then there exists $\delta > 0$ such that $f(x) < g(x)$ for all $x \in (c - \delta, c + \delta)$.

51. Suppose that f has an essential discontinuity at c. Change the value of f as you choose at any finite number of points $x_1, x_2, \ldots, x_n$ and call the resulting function g. Show that g also has an essential discontinuity at c.

52. (a) Prove that if f is continuous everywhere, then $|f|$ is continuous everywhere.
(b) Give an example to show that the continuity of $|f|$ does not imply the continuity of f.
(c) Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

53. Suppose the function f has the property that there exists a number B such that

$$|f(x) - f(c)| \leq B|x - c|$$

for all x in the interval $(c - p, c + p)$. Prove that f is continuous at c.

54. Suppose the function f has the property that

$$|f(x) - f(t)| \leq |x - t|$$

for each pair of points x, t in the interval (a, b). Prove that f is continuous on (a, b).

55. Prove that if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then f is continuous at c.

56. Suppose that the function f is continuous on $(-\infty, \infty)$. Show that f can be written

$$f = f_e + f_0,$$

where f_e is an even function which is continuous on $(-\infty, \infty)$ and f_0 is an odd function which is continuous on $(-\infty, \infty)$.

Exercises 57–60. The function f is not defined at $x = 0$. Use a graphing utility to graph f. Zoom in to determine whether there is a number k such that the function

$$F(x) = \begin{cases} f(x), & x \neq 0 \\ k, & x = 0 \end{cases}$$

is continuous at $x = 0$. If so, what is k? Support your conclusion by calculating the limit using a CAS.

57. $f(x) = \dfrac{\sin 5x}{\sin 2x}$.

58. $f(x) = \dfrac{x^2}{1 - \cos 2x}$.

59. $f(x) = \dfrac{\sin x}{|x|}$.

60. $f(x) = \dfrac{x \sin 2x}{\sin x^2}$.