Problem 7: Show $y_n = \sum_{k=0}^{2n+1} \frac{(-1)^k}{.....}$ converges, with its limit irrational.

*Proof.* Note that

$$y_n = 1 - \frac{1}{2!} + \frac{1}{4!} + ... + \frac{1}{(2(2n))!} - \frac{1}{(2(2n+1))!}$$

We see that

$$y_{n+1} - y_n = \frac{1}{(2(2n+2))!} - \frac{1}{(2(2n+3))!} > 0$$

Moreover, according to textbook and notes on courses,

$$\begin{aligned} y_n &\leq 1 + \frac{1}{2!} + \frac{1}{4!} + ... + \frac{1}{(4n)!} + \frac{1}{(4n+2)!} \\ &< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{(4n+2)!} < 3 \end{aligned}$$

Hence $(y_n)$ is convergent (to $\cos 1$). Denote the limit by $y$. For $n$,

$$\begin{aligned} &|y - y_n| = \frac{1}{(4n+4)!} - \frac{1}{(4n+6)!} + .... \\ &\Rightarrow |((4n)!) \cdot y - ((4n)! y_n)| \\ &\quad \leq \frac{1}{4n+4} + \frac{1}{(4n+4)(4n+5)} + \frac{1}{(4n+4)(4n+5)(4n+6)} + ... \\ &\quad \leq \frac{1}{4n+4}(1 + \frac{1}{4n+5} + \frac{1}{(4n+5)(4n+6)} + \frac{1}{(4n+6)(4n+7)} + ...) \\ &\quad < \frac{1}{4n+4}(1 + \frac{3}{4n+5}) < \frac{4}{4n+4} = \frac{1}{n+1} \to 0 \end{aligned}$$

By theorem, $y$ is irrational.

$\square$