Problem 5: $\sum a_j = 0$. Prove $\lim a_0\sqrt{n} + ... + a_k\sqrt{n+k} = 0$.

Proof. Approximating the error

$$
\begin{aligned}
&|a_0\sqrt{n} + a_1\sqrt{n+1} + ... + a_k\sqrt{n+k}| \\
&= |(-a_1 - ... - a_k)\sqrt{n} + a_1\sqrt{n+1} + ... + a_k\sqrt{n+k}| \\
&\le |a_1||\frac{1}{\sqrt{n+1}+\sqrt{n}}| + |a_2||\frac{2}{\sqrt{n+2}+\sqrt{n}}| + ... + |a_k||\frac{k}{\sqrt{n+k}+\sqrt{n}}| \\
&\le M \cdot |\frac{1}{\sqrt{n+1}+\sqrt{n}}| \le \frac{M}{\sqrt{n}}
\end{aligned}
$$

where $M = k^2 \cdot Max\{|a_j| : j = 1, 2, ..., k\}$. Let ε>0. Choose $N_\varepsilon = [\frac{M^2}{\varepsilon^2}] + 10$. Then as $n \ge N_\varepsilon$,

$$
|a_0\sqrt{n} + a_1\sqrt{n+1} + ... + a_k\sqrt{n+k}| \le \frac{M}{\sqrt{n}} \le \frac{M}{\sqrt{N_\varepsilon}} < \frac{M}{\sqrt{\frac{M^2}{\varepsilon^2}}} = \varepsilon
$$

□

[Note] The key point is: (i) To properly use the condition $\sum a_j = 0$ to rewrite the term in $\lim_{n\to\infty}$, and (ii) to approximate the error of the sum of several terms (we are assumed to have no arithmetic properties on limits).