6. Show $\alpha_n := \prod_{j=1}^{n} 1 - \frac{1}{3^j}$ converges, with it limit positive.

[Solution] It' trivial that (α_n) decreases, with 0 its lower bound. It converges. I'll give two ways to show that it has a positive lower bound.

(Method I) We know that $(1-a)(1-b)>1-(a+b)$ for $a, b \in (0,1)$. This means

$$
\begin{aligned}
&(1-\frac{1}{3^1})(1-\frac{1}{3^2})(1-\frac{1}{3^3})...(1-\frac{1}{3^n})\\
&>[1-(\frac{1}{3^1}+\frac{1}{3^2})](1-\frac{1}{3^3})...(1-\frac{1}{3^n})\\
&>[1-(\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3})]...(1-\frac{1}{3^n})\\
&>...>1-(\frac{1}{3^1}+\frac{1}{3^2}+...+\frac{1}{3^n})\\
&=1-\frac{1}{3}\frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}>\frac{1}{2}
\end{aligned}
$$

So we conclude that the limit might greater than 0.

(Method II) By an inequality $3^n>n^2$, we find that $1-\frac{1}{3^j}>1-\frac{1}{j^2}$. Hence

$$
\begin{aligned}
&\prod_{j=1}^{n} 1-\frac{1}{3^j}>\frac{2}{3}\prod_{j=2}^{n} 1-\frac{1}{j^2}=\prod_{j=1}^{n}(1-\frac{1}{j})(1+\frac{1}{j})\\
&=\frac{2}{3}(\frac{1}{2}\frac{2}{3}...\frac{n-1}{n})(\frac{3}{2}\frac{4}{3}...\frac{n+1}{n})\\
&=\frac{n+1}{3n}>\frac{n}{3n}=\frac{1}{3}
\end{aligned}
$$

Similarly we finish the proof.