Problem 3. Discuss convergence of $\alpha_n := \frac{1}{n^5}\sum_{k=1}^{n} k^4$.

[Solution] (1) We know that

$$x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

Then

$$\begin{aligned} 2^5 - 1^5 = &\quad 2^4 + 2^3 \cdot 1 + 2^2 \cdot 1^2 + 2 \cdot 1^3 + 1^4 \\ 3^5 - 2^5 = &\quad 3^4 + 3^3 \cdot 2 + 3^2 \cdot 2^2 + 3 \cdot 2^3 + 2^4 \\ &\vdots \\ (n+1)^5 - n^5 = &\quad (n+1)^4 + (n+1)^3 \cdot n + (n+1)^2 \cdot n^2 + (n+1) \cdot n^3 + n^4 \end{aligned}$$

From all above, we find that

$$\begin{aligned} (n+1)^5 - 1 &= \sum (k+1)^4 + \sum (k+1)^3 k + \sum (k+1)^2 k^2 + \sum (k+1) k^3 + \sum k^4 \\ &= 5\sum k^4 + 10 \sum k^3 + 10 \sum k^2 + 5 \sum k + n \end{aligned}$$

After simplification, we get

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

To show (α_n)'s convergence. We guess the limit is $\frac{1}{5}$. Estimating the error

$$|\alpha_n - \frac{1}{5}| = \frac{1}{n}(\frac{1}{2} + \frac{1}{3n} - \frac{1}{30n^3}) < \frac{1}{n} \cdot (\frac{1}{2} + \frac{1}{3} + 0) = \frac{5}{6}\frac{1}{n}.$$

Given $\varepsilon > 0$, choose $N_\varepsilon = [\frac{5}{6\varepsilon}] + 1$. Then when $n \geq N_\varepsilon$,

$$|\alpha_n - \frac{1}{5}| < \frac{5}{6}\frac{1}{n} \leq \frac{5}{6}\frac{1}{N_\varepsilon} \leq \frac{5}{6}\frac{1}{\frac{5}{6\varepsilon}} = \varepsilon$$

Hence (α_n) converges to $\frac{1}{5}$.

[Note] I prefer another way for computation.

$$\begin{aligned} (1+1)^5 &= 1^5 + 5 \cdot 1^4 + 10 \cdot 1^3 + 10 \cdot 1^2 + 5 \cdot 1 + 1 \\ (2+1)^5 &= 2^5 + 5 \cdot 2^4 + 10 \cdot 2^3 + 10 \cdot 2^2 + 5 \cdot 2 + 1 \\ &\vdots \\ (n+1)^5 &= n^5 + 5 \cdot n^4 + 10 \cdot n^3 + 10 \cdot n^2 + 5 \cdot n + 1 \end{aligned}$$

Then summing them,

$$(n+1)^5 = 1 + 5\sum k^4 + 10\sum k^3 + 10 \sum k^2 + 5 \sum k + n$$

and we will get the same sum.

Problem 4. Does $g_n := \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}k$ converge?

[Solution] We compute some terms:

$$1, -\frac{1}{2}, \frac{2}{3}, -\frac{1}{2}, \frac{3}{5}, -\frac{1}{2}, \ldots$$

i.e.

$$g_{2k-1} = \frac{k}{2k-1}$$
$$g_{2k} = -\frac{1}{2}.$$

But we find that $(g_{2k-1})_{k=1}^{\infty}$ goes to $\frac{1}{2}$ while $(g_{2k})_{k=1}^{\infty}$ to $-\frac{1}{2}$. Hence (g_n) diverges.

[Note] There is a corollary saying that any subsequence of a convergent sequence must converge to the limit as the original sequence do. This indicates a good way to test a divergent sequence: to find two subsequences converging to different limits.