1. (a) Show that $a_n \geq n$. (b) Find N s.t. $|\frac{a_n}{a_n+1} - 1|<\varepsilon$.

[Solution] (a) We prove it by induction. Since a_1 is a positive integer, we know that $a_1 \geq 1$. Assume that $k \in \mathbb{Z}_+$ and $a_k \geq k$. Since (a_n) is a strictly increasing sequence of positive integers, we get $a_{k+1} - a_k \geq 1$. So

$$a_{k+1} = (a_{k+1} - a_k) + a_k \geq 1 + k = k + 1$$

Therefore, induction tells us that $a_n \geq n$ for positive n.

(b) We are asked to solve the inequality $|\frac{a_n}{a_n+1} - 1|<\varepsilon$. The solution is

$$a_n>\frac{1}{\varepsilon} - 1 \tag{1}$$

We hope to find suffuciently large n's satisfying (1). It suffices to make n large enough so that

$$n>\frac{1}{\varepsilon} - 1$$

We might choose $N_\varepsilon = [\frac{1}{\varepsilon}] + 1$. Then for $n \geq N_\varepsilon$, by (a) we find that

$$a_n \geq n \geq N_\varepsilon = [\frac{1}{\varepsilon}] + 1 \geq \frac{1}{\varepsilon}>\frac{1}{\varepsilon} - 1$$
$$\Rightarrow -\varepsilon<0<\frac{1}{a_n + 1}<\varepsilon$$

Hence, this N_ε does work.

[Note]

By (a) we're expected to use the form $a_{k+1} = (a_{k+1} - a_k) + a_k$, and to be familiar with properties about positive integers, i.e. the fact that $|n-m| \geq 1$ whenever $n \neq m$. The proof by mathematical induction is an ideal way due to preciseness.

(b) Many people might find that we have no idea about what a_n is at the first sight, but part (a) is a helpful message. In fact, we are suggested to solve it on a_n, and then find relations between a_n and n.

2. Show $|\sin(nx)| \leq n \sin x$.

[Solution] According to the expression $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Substititing nx to θ, we get

$$\begin{aligned}|\sin(nx)| &= |\frac{e^{inx}-e^{-inx}}{2i}| = |\frac{(e^{ix})^n-(e^{-ix})^n}{2i}| \\ &= |\frac{e^{ix}-e^{-ix}}{2i}||(e^{ix})^{n-1}+(e^{ix})^{n-1}(e^{-ix})^1+....+(e^{-ix})^{n-1}| \\ &\leq \sin x\cdot(|(e^{ix})^{n-1}|+|(e^{ix})^{n-1}(e^{-ix})^1|+....+|(e^{-ix})^{n-1}|) \\ &= \sin x\cdot(1+1+...+1) = n\sin x\end{aligned}$$

[note] The key point is to properly use the exponential expression, and to factorize. What to prove will emerage after application of triangular inequality. [Note] If you think the "notation" e^{ix} cumbersome, you might as well alternatively define the function $e(x)$ by

$$e(x) = \cos x + i\sin x$$

Then we have (i) $(e(x))^n = e(nx)$, and (ii) $\sin\theta = \frac{e(x)-e(-x)}{2i}$. Hence we may do the same thing to prove this problem without application of "complex exponentiation".