Problem 6: Let $(s_n)_{n=1}^\infty$ be a convergent sequence with limit S. Then for any $\epsilon > 0$, there exists an index N_ϵ, such that $S - \epsilon < s_n < S + \epsilon$ for all $n \geq N_\epsilon$. Using $(s_n)_{n=1}^\infty$ to define a new sequence $(\sigma_n)_{n=1}^\infty$ by letting $\sigma_n = \frac{s_1 + \cdots + s_n}{n}$.

(1) Prove that for any $\epsilon > 0$ there exists an index N_ϵ, such that for all $n > N_\epsilon$, the following inequality holds:

$$\frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S - \varepsilon)}{n} < \sigma_n < \frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S + \varepsilon)}{n}.$$

(2) Prove that the sequence $(\sigma_n)_{n=1}^\infty$ is also convergent, and its limit is S.

(3) Define $\sigma_1 = 1, \sigma_n = 1 + (1 - \frac{1}{n})\frac{1}{2} + (1 - \frac{2}{n})\frac{1}{2^2} + ... + (1 - \frac{n-1}{n})\frac{1}{2^{n-1}}$, for $n \geq 2$. Prove that $(\sigma_n)_{n=1}^\infty$ is convergent, nd find its limit.

[Solution] (1) Let $\varepsilon > 0$. By definition there is an N_ε such that for all $n > N_\varepsilon$,

$$s_1 = s_1 = s_1$$

$$\vdots$$

$$s_{N_\varepsilon} = s_{N_\varepsilon} = s_{N_\varepsilon}$$
$$S - \varepsilon < s_{N_\varepsilon + 1} < S + \varepsilon$$

$$\vdots$$

$$S - \varepsilon < s_n < S + \varepsilon$$

$$(1)$$

Summing up, we get

$$\frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S - \varepsilon)}{n} < \sigma_n < \frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S + \varepsilon)}{n}.$$

(2)

Proof. [1] Let $0 < \epsilon' < 1$. Then by (1), we find an $N_{\epsilon'}$ such that $\forall n \geq N_{\epsilon'}$,

$$\frac{s_1 + .. + s_{N_{\epsilon'}}}{n} + \frac{(n - N_{\epsilon'})(S - \epsilon')}{n} < \sigma_n < \frac{s_1 + .. + s_{N_{\epsilon'}}}{n} + \frac{(n - N_{\epsilon'})(S + \epsilon')}{n}.$$

Since $\lim_{n \to \infty} \frac{s_1 + .. + s_{N_{\epsilon'}}}{n} = 0$ and $\lim_{n \to \infty} \frac{n - N_{\epsilon'}}{n} = 1$, there is an $N'_{\epsilon'} > N_{\epsilon'}$ such that for all $n \geq N'_{\epsilon'}$,

$$-\epsilon' < \frac{s_1 + ... + s_{N_{\epsilon'}}}{n} < \epsilon'$$
$$1 - \epsilon' < \frac{n - N_{\epsilon'}}{n} < 1 + \epsilon'$$

At this time,

$$\sigma_n < \epsilon' + (1 \pm \epsilon')(S + \epsilon')$$
$$= \epsilon' + S + (1 \pm S)\epsilon' \pm \epsilon'^{\,2}$$
$$\leq \epsilon' + S + |1 \pm S|\epsilon' + \epsilon'$$
$$\leq \epsilon' + S + (1 + |S|)\epsilon' + \epsilon'$$
$$= S + (3 + |S|)\epsilon'$$

$$\sigma_n > -\epsilon' + (1 \mp \epsilon')(S - \epsilon')$$
$$= -\epsilon' + S - (1 \pm S)\epsilon' \pm \epsilon'^{\,2}$$
$$\geq -\epsilon' + S - |1 \pm S|\epsilon' - \epsilon'$$
$$\geq -\epsilon' + S - (1 + |S|)\epsilon' - \epsilon'$$
$$= S - (3 + |S|)\epsilon'$$

where the choice of $\pm$ and $\mp$ depends on the sign of $S + \epsilon'$ and $S - \epsilon'$. Hence

$$|\sigma_n - S| < (3 + |S|)\epsilon'$$

[2] Next, we prove $\lim_{n \to \infty} \sigma_n = S$. Let $\varepsilon > 0$. Choose $\epsilon' = min\{\frac{\varepsilon}{3+|S|}, \frac{1}{2}\}$. Then there is an $N_{\epsilon'}$ and an $N'_{\epsilon'} > N_{\epsilon'}$, such that for any $n \geq N'_{\epsilon'}$,

$$|\sigma_n - S| < (3 + |S|)\epsilon'$$
$$\leq (3 + |S|)\frac{\varepsilon}{3 + |S|} = \varepsilon$$

$\square$

[Note] (a) The limit $\lim_{n \to \infty} \sigma_n$ is call *Cesaro Sum*. (b) If we have the technique of upper and lower limits, the proof can be simplified.

(3) Let $\tau_n = 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}}$, $\varsigma'_n = \frac{1}{n}\{0 \cdot \frac{1}{2^0} + 1 \cdot \frac{1}{2^1} + ... + (n-1)\frac{1}{2^{n-1}}\}$, $s_n = \frac{n-1}{2^{n-1}}$. Then

$$\sigma_n = \tau_n - \varsigma'_n$$

Since $s_n \to 0$ (check !), we get $\varsigma'_n \to 0$. Since we also have $\tau_n \to 2$, then $\sigma_n \to 2$ as well.