Problem 6: (1) The inequality. (2)Show that $(\sigma_n) \to S$. (3) Evaluate a limit.
[Solution] (1) Let $\varepsilon > 0$. By definition there is an N_ε such that for all $n > N_\varepsilon$,

$$\begin{aligned} s_1 &= s_1 = s_1 \\ &\vdots \\ s_{N_\varepsilon} &= s_{N_\varepsilon} = s_{N_\varepsilon} \\ S - \varepsilon &< s_{N_\varepsilon+1} < S + \varepsilon \\ &\vdots \\ S - \varepsilon &< s_n < S + \varepsilon \end{aligned} \tag{1}$$

Summing up, we get

$$\frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S - \varepsilon)}{n} < \sigma_n < \frac{s_1 + .. + s_{N_\varepsilon}}{n} + \frac{(n - N_\varepsilon)(S + \varepsilon)}{n}.$$

(2)

Proof. [1] Let $0 < \epsilon' < 1$. Then by (1), we find an $N_{\epsilon'}$ such that $\forall n \geq N_{\epsilon'}$,

$$\frac{s_1 + .. + s_{N_{\epsilon'}}}{n} + \frac{(n - N_{\epsilon'})(S - \epsilon')}{n} < \sigma_n < \frac{s_1 + .. + s_{N_{\epsilon'}}}{n} + \frac{(n - N_{\epsilon'})(S + \epsilon')}{n}.$$

Since $\lim_{n\to\infty} \frac{s_1+..+s_{N_{\epsilon'}}}{n} = 0$ and $\lim_{n\to\infty} \frac{n-N_{\epsilon'}}{n} = 1$, there is an $N'_{\epsilon'} > N_{\epsilon'}$ such that for all $n \geq N'_{\epsilon'}$,

$$\begin{aligned} -\epsilon' &< \frac{s_1 + ... + s_{N_{\epsilon'}}}{n} < \epsilon' \\ 1 - \epsilon' &< \frac{n - N_{\epsilon'}}{n} < 1 + \epsilon' \end{aligned}$$

At this time,

$$\begin{aligned} \sigma_n &< \epsilon' + (1 \pm \epsilon')(S + \epsilon') \\ &= \epsilon' + S + (1 \pm S)\epsilon' \pm \epsilon'^{\ 2} \\ &\leq \epsilon' + S + |1 \pm S|\epsilon' + \epsilon' \\ &\leq \epsilon' + S + (1 + |S|)\epsilon' + \epsilon' \\ &= S + (3 + |S|)\epsilon' \end{aligned}$$

$$\begin{aligned} \sigma_n &> -\epsilon' + (1 \mp \epsilon')(S - \epsilon') \\ &= -\epsilon' + S - (1 \pm S)\epsilon' \pm \epsilon'^{\ 2} \\ &\geq -\epsilon' + S - |1 \pm S|\epsilon' - \epsilon' \\ &\geq -\epsilon' + S - (1 + |S|)\epsilon' - \epsilon' \\ &= S - (3 + |S|)\epsilon' \end{aligned}$$

where the choice of $\pm$ and $\mp$ depends on the sign of $S+\epsilon'$ and $S-\epsilon'$. Hence

$$|\sigma_n - S|<(3+|S|)\epsilon'$$

[2] Next, we prove $\lim_{n\to\infty}\sigma_n = S$. Let $\varepsilon>0$. Choose $\epsilon' = min\{\frac{\varepsilon}{3+|S|}, \frac{1}{2}\}$. Then there is an $N_{\epsilon'}$ and an $N'_{\epsilon'}>N_{\epsilon'}$, such that for any $n \geq N'_{\epsilon'}$,

$$\begin{aligned} |\sigma_n - S| &< (3+|S|)\epsilon' \\ &\leq (3+|S|)\frac{\varepsilon}{3+|S|} = \varepsilon \end{aligned}$$

□

[Note] (a) The limit $\lim_{n\to\infty}\sigma_n$ is call *Cesaro Sum*. (b) If we have the technique of upper and lower limits, the proof can be simplified.

(3) Let $\tau_n = 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}}$, $\varsigma'_n = \frac{1}{n}\{0\cdot\frac{1}{2^0} + 1\cdot\frac{1}{2^1} + ... + (n-1)\frac{1}{2^{n-1}}\}$, $s_n = \frac{n-1}{2^{n-1}}$. Then

$$\sigma_n = \tau_n - \varsigma'_n$$

Since $s_n \to 0$ (check !), we get $\varsigma'_n \to 0$. Since we also have $\tau_n \to 2$, then $\sigma_n \to 2$ as well.