Problem 2. (1)(2) Identities. ($2\frac{1}{2}$) $q_n \geq n$. (3)(4) Odd terms and even terms converge. (5) The whole sequence converges.

[Solution] (1)(2) We know that

$$\begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} \cdot \begin{pmatrix} a_k+1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k a_{k+1} + p_{k-1} & p_k \\ q_k a_{k+1} + q_{k-1} & q_k \end{pmatrix}$$

This means

$$p_{k+1} = p_k a_{k+1} + p_{k-1}$$
$$q_{k+1} = q_k a_{k+1} + q_{k-1}$$

By computation, and these two formulae, we have

$$p_{k+1}q_k - q_{k+1}p_k = -(p_k q_{k-1} - q_k p_{k-1})$$
$$p_{k+1}q_{k-1} - q_{k+1}p_{k-1} = a_{k+1}(p_k q_{k-1} - q_k p_{k-1})$$

Inductively we get

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$$
$$p_{n+2}q_n - q_{n+2}p_n = a_{n+2}(p_{n+1}q_n - q_{n+1}p_n) = a_{n+2}(-1)^n$$

($2\frac{1}{2}$) $q_1 = a_1 q_0 + q_{-1} \geq 1 \cdot 1 + 0 = 1$. $q_2 = a_2 q_1 + q_0 \geq 1 \cdot 1 + 1 = 2$. If $q_k \geq k$ and $q_{k+1} \geq k+1$, then

$$q_{k+2} = a_{k+2}q_{k+1} + q_k \geq 1 \cdot (k+1) + k \geq k+2$$

Hence $q_n \geq n$.

(3)(4) By the second part,

$$\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n} = (-1)^n \frac{a_{n+2}}{q_{n+2}q_n}$$

Since a_{n+2}, q_{n+2}, and $q_n \in \mathbb{N}$, we prove that both sequence are decreasing and increasing respectively. By the first part we find

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_n q_{n-1}} \leq \frac{(-1)^{n+1}}{(n-1)^2} \qquad (*)$$

If $k<l$, then

$$\frac{p_1}{q_1} > \frac{p_{2k+1}}{q_{2k+1}} > \frac{p_{2l+1}}{q_{2l+1}} > \frac{p_{2l}}{q_{2l}} > \frac{p_{2k}}{q_{2k}} > \frac{p_2}{q_2}$$

Hence both sequences are bounded, and then convergent.

(5) By $(*)$, we find that $\lim_{n\to\infty} \frac{p_{2k+1}}{q_{2k+1}} - \frac{p_{2k}}{q_{2k}} = 0$. Since both limits exist, they are equal. Zipper theorem then yields that $(\frac{p_n}{q_n})$ converges.