Problem 4: (a_n) bdd. All a_{n_j} converges to a common α. Does (a_n) converge??

Proof. Assume that (c_n) doesn't converge to α, the common limit of all convergent subsequences. Then there is an $\varepsilon_0>0$ such that for all $N \in \mathbb{N}$, there is a corresponding term $n_N>N$ such that $|a_{n_N} - \alpha| \geq \varepsilon_0$.

Now, we form a subsequence

$$c_{n_1}, \quad c_{n_{n_1}}, \quad c_{n_{n_{n_1}}}, \cdots$$

and denote it by $(b_j)_{j=1}^{\infty}$. If (b_j) contains infinitely many terms such that

$$b_j \geq \alpha + \varepsilon_0, \tag{1}$$

then, we ignore the opposite terms and form a subsequence $(b_{j_k})_{k=1}^{\infty}$. If (b_j) contains only finitely many terms such that (1) holds, then we ignore these terms and keep the opposite terms, forming the subsequence $(b_{j_k})_{k=1}^{\infty}$. In these cases, we have $b_{j_k} \geq \alpha + \varepsilon_0$ and $b_{j_k} \leq \alpha - \varepsilon_0$ respectively. Bolzano-Weierstrass Theorem yields that we have a convergent subsequence

$$(b_{j_{k_l}})_{l=1}^{\infty}$$

and its limit must $\geq \alpha + \varepsilon_0 > \alpha$ ($\leq \alpha - \varepsilon_0 < \alpha$ respectively). But condition says that the limit equals α, a contradiction. □

[Note] This exercise indicates that if a bounded sequence (a_n) has merely one accumulation point a, then $a_n \to a$. But if not bounded, for example, $a_{2k+1} = (-k)^k$ while $a_{2k} = (-\frac{1}{k})^k$, this statement is not necessarily true.