Why do we need a proof of 1+1=2

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Reliability is the essence of a proof. This means, to give a proof of a fact, we want to pursuade ourselves its credibility. Everything comes with reasons, and of course, including thoughts of Mathematics.

Many people might have heard of that there's a proof of the fomula

$$1 + 1 = 2.$$

At the first sight, nearly all people would think that it must be insane to prove that, because it really has nothing to do. We can just put an apple and another one together, then these two apples is the answer.

This is a reasonable reply. Indeed, without further observation on higher mathematics, the requirement (or, the desire) of a "proof" would not emerge.

Theory of mathematics always develops according to our practical needs. For example, by counting apples, we have

$$1 + 1 = 2;$$

by dividing a pie, we have

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6};$$

the theorem of right triangles leads to discovery of the irrational number, $\sqrt{2}$, and then we have

$$\frac{1}{\sqrt{2}+1} + 1 = \sqrt{2}.$$

Operations on real numbers follows quite from the visualization of the real line, which contains only one axis. With a connection to classical geometry, the introduction of rectangular coordinate in some sense is essential.

Moreover, we can represent graph of functions on rectangular coordinate, and investigate more properties.

Limits, continuity, slope of a tangent line, and area under a curve, are important geometric topics, with limits their bases. However, disastrous consequences occurrs during the process. For example, we have

$$0 = \lim_{y\to 0}\lim_{x\to 0}\frac{x}{x+y} = \lim_{x\to 0}\lim_{y\to 0}\frac{x}{x+y} = 1.$$

The deduction seems to make sense because that x goes to 0, and then y goes to 0 means that x,y both go to 0, which also means that y goes to 0, and then x goes to 0. However, the result that $0 = 1$ is absurd. There might be something wrong within it, and indeed, the key point is that we cannot change the order of the limit notations.

The main problem is that the meaning of the limit is not clear enough so that we misreorder the two limits. A formal definition required is:

$$\lim_{x\to c} f(x) = L \Leftrightarrow_{Def} (\forall\varepsilon{>}0)(\exists\delta{>}0)(\text{if } 0{<}|x-c|{<}\delta, \text{ then}|f(x)-L|{<}\varepsilon)$$

This is an illustration of the notion of limits. If we want to check the truch of

$$\lim_{x\to 2} 3x - 5 = 1,$$

this is a proof: Let $\varepsilon{>}0$. We choose $\delta = \frac{\epsilon}{3}$. Then as $0{<}|x-2|{<}\delta$,

$$|(3x-5)-1| = 3|x-2|{<}3\delta = 3\frac{\varepsilon}{3} = \varepsilon$$

What I have to mention is that, from now on the truth of terms involving limits depends only on the right hand side of above definition —- which involves only operations. This means, if we want to verify whether the limit of a term is the given number, we must have been familiar with properties of real numbers.

Hence another problem arises —— since we may accidentally make mistakes when dealing with problems concerning "unusual" limits, i.e. there are many traps in the world of limits, however, is there any trap in real numbers? Will we "stumble" again while struggling with real numbers? To make our question more precise, are $|3x-6| = 3|x-2|$, and $3\frac{\varepsilon}{3} = \varepsilon$ reliable?

We will look back upon properties on real numbers. Our problem is that, how arithmetric properties of real number are built. Unfortunately, they are presently constructed by, only intuition. If we agree that the above definition of limit is stable enough, then there must be a way to construct the real number system like this.

In 19th century, one of the most famous definition of real numbers, is done by Dedekind, though Dedekind cuts. A Dedekind Cut ξ is a subset of the rational number set $\mathbb{Q}$, such that

[a]ξ is nonempty and is not $\mathbb{Q}$ itself.

[b]If $t \in \xi$, then for any rational number $s<t$, $s \in \xi$

[c]For any $t \in \xi$, there is a rational number $s>t$ such that $s \in \xi$

The background idea is that we can cut the "rational line" (with respect to the notion of real line), into two part, and the part in our left hands is a Dedekind cut, and will represent a real number, For example, the cut $\xi = \{t \in \mathbb{Q}^+ | t^2<2\} \cup \mathbb{Q}^-$.

Addition and multiplication on two Dedekind cuts ξ, η is defined as the cut

$$\xi + \eta := \{u + v | u \in \xi, v \in \eta\}$$
$$\xi \cdot \eta := \{u \cdot v | u \in \xi, v \in \eta\}$$

We might feel that, construction of the theory of real numbers without intuition involves a lot of properties of rational numbers, For example, the equality that

$$\{t \in \mathbb{Q}^+ | t^2<2\} \cdot \{t \in \mathbb{Q}^+ | t^2<3\} = \{t \in \mathbb{Q}^+ | t^2<6\}.$$

Our target has been transfered to the theory of rational numbers. It seems to be easier to explain the identities $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$, $2 = \frac{2}{1} = \frac{4}{2}$, $a \cdot \frac{2}{a} = 2$, and so on. In fact, rational numbers are defined as "equivalence classes" of a given pair. Briefly speaking, for example.

$$
\begin{aligned}
&a/b := \{\langle x,y\rangle | \langle x,y\rangle \sim \langle a,b\rangle\}\\
&Then\\
&1/2 = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, ...\}\\
&2/4 = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, ...\}\\
&3/6 = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, ...\}
\end{aligned} \tag{1}
$$

Collecting those "(a/b)'s", this is the rational number set. Moreover, we denote $\bar{1}$ for $1/1$, $\bar{2}$ for $2/1$, and so on, but for writing convenience, we always omit the "bar". Operations on rationals are defined in the natural way. However, it's clear that $2 = \frac{2}{1} = \frac{4}{2}$ remains unsolved. Indeed, we still need something improve it.

There is a theorem saying that for rational numbers X, and $Y \neq 0$, there is exactly one rational U such that

$$X \cdot U = Y$$

This U is denoted by "$\frac{Y}{X}$". Then $2 = \frac{2}{1} = \frac{4}{2}$ becomes trivial. (In fact, this identity is abbreviated. The initial form is that

$$\frac{\bar{2}}{\bar{1}} =_{Thm} \bar{2} (=_{Def} 2/1 = 4/2) = \frac{\bar{4}}{\bar{2}}.$$

Also, $a \cdot \frac{2}{a} = 2$ would make sense.

When we want to prove that whenever $0 < A < B$, and $0 < C < D$, we always have that $0 < AC < BD$, we ought to have known that of natural number version. The reason is, after expressing $A = \frac{a}{\alpha}$, $B = \frac{b}{\beta}$, $C = \frac{c}{\gamma}$, and $D = \frac{d}{\delta}$, we get from conditions that

$$
\begin{aligned}
a\beta &< \alpha b\\
c\delta &< \gamma d,
\end{aligned}
$$

then, according to properties of natural numbers, we infer

$$
\begin{aligned}
(ac)(\beta\delta) &< (bd)(\alpha\gamma)\\
AC &< BD
\end{aligned}
$$

Now, our problem is reduced to integers and natural numbers. To be more elementary, we want to know operations on them, e.g. that

$$(1+2)+3 = 1+(2+3)$$
$$4+7 = 7+4$$

Moreover, we are even led to consideration of the truth of the formula that

$$1+1=2.$$

What we ought to illustrate is what 1,+,=,2, really are.

I don't prepare to give an answer to this problem. A reference of this (constructions of number systems) is: *Foundations of analysis*, written by the German mathematician, *Edmund Landau*. During the reading of this books, we could solve all problems listed above. However, if you are sensitive on disadvantage of the discuss, and insist in what is called formalism of foundations of mathematics, then you will face **a series of** problems.