Meaningful Products

Wang-Shiuan Pahngerei

Let G be a semigroup and A a set. Denote $\bar{G} = G \times \wp A$. Then the operation

$$\begin{pmatrix} g_1 \\ A_1 \end{pmatrix} * \begin{pmatrix} g_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 \\ A_1 \cup A_2 \end{pmatrix}$$

gives a structure of a semigroup. Let $\alpha : |1, N| \to G$, written multiplicatively, and $\beta : |1, N-1| \to \mathbb{N}$ be such that $\beta(n) \leq N - n$ for all n. Set $A = |1, N|$. Define $S : |0, N-1| \to$ {functions from a subset of $|1, N|$ to $\bar{G}$ } be a sequence such that
$S(0) = \left(\begin{pmatrix} \alpha(1) \\ \{1\} \end{pmatrix}, \begin{pmatrix} \alpha(2) \\ \{2\} \end{pmatrix} \cdots, \begin{pmatrix} \alpha(N) \\ \{N\} \end{pmatrix} \right)$.
$S(1) = (S_0(1), S_0(2), \cdots, S_0(\beta_1) * S_0(\beta_1 + 1), \cdots, S_0(N))$, with $N - 1$ terms.
$S(2) = (S_1(1), \cdots, S_1(\beta_2) * S_1(\beta_2 + 1), \cdots, S_1(N-1))$, with $N - 2$ terms.
$\vdots$
$S(k+1) = (S_k(1), \cdots, S_k(\beta_{k+1}) * S_k(\beta_{k+1} + 1), \cdots, S_k(N-k))$.

Note that [1] each $S(j)(k)(2)$ is a consecutive set, i.e. the set $|p, q|$. [2] Then $S(N-1) : \{1\} \to \bar{G}$. At this time we define $\prod_\beta \alpha = S(N-1)(1)(1)$, i.e. $S(N-1) = \left(\begin{pmatrix} \prod_\beta \alpha \\ * \end{pmatrix} \right)$.

We want to show for different β and β', $\prod_\beta \alpha = \prod_{\beta'} \alpha$. Intuitively,

$$\begin{aligned} &((((a_1 + a_2) + (a_3 + a_4)) + a_5) + (a_6 + a_7)) + (a_8 + (a_9 + a_{10})) \\ &\qquad = ((a_1 + a_2) + (a_3 + (a_4 + a_5))) + ((a_6 + a_7) + ((a_8 + a_9) + a_{10})) \end{aligned}$$

Sketch of proof. By induction on $N \geq 3$. In case it holds at n, then for any two different orders of summations,

$$\begin{aligned} &(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n+1}) \\ &= ((a_1 + \cdots + a_\ell) + (a_{\ell+1} + \cdots + a_k)) + (a_{k+1} + \cdots + a_{n+1}) \\ &= (a_1 + \cdots + a_\ell) + ((a_{\ell+1} + \cdots + a_k) + (a_{k+1} + \cdots + a_{n+1})) \\ &= (a_1 + \cdots + a_\ell) + ((a_{\ell+1} + \cdots + a_{n+1})) \end{aligned}$$

The first and third equalities follow from induction hypothesis while the second from associativity. □

At this time, we define for a sequence $\alpha = (\alpha_1, \cdots, \alpha_N)$ its product $\prod \alpha$ by $\prod_{(1,1,\cdots,1)} \alpha$.