I think that exponentiation is a topic which most of us are familiar with, but however when considering some specific problems, we might find that we get into a dilemma, For example, the equalities

$$-1 = \sqrt[3]{-1} = (-1)^{\frac{1}{3}} = (-1)^{\frac{2}{6}} = \sqrt[6]{(-1)^2} = \sqrt[6]{1} = 1 \tag{1}$$

$$\begin{aligned} 1 &= \cos 2\pi + i \sin 2\pi = [(\cos 2\pi + i \sin 2\pi)^2]^{\frac{1}{2}} = (\cos 4\pi + i \sin 4\pi)^{\frac{1}{2}} \\ &= (\cos 2\pi + i \sin 2\pi)^{\frac{1}{2}} = [(\cos \pi + i \sin \pi)^2]^{\frac{1}{2}} = cos\pi + i \sin \pi = -1 \end{aligned} \tag{2}$$

There are classic misuses of properties of exponentiation. This leads us to a goal whcih is, what exactly the rules are ? Now, it is time for us to rebuild such a system (that of the exponentiation). Our work focus on the definitions of each type of exponentiation, and their laws.

Firstly, by induction, we define that for $a \in \mathbb{R}$ (or $\mathbb{C}$), and $n \in N$,

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n \times a \qquad \text{for } n \geq 1 \end{aligned}$$

Briefly speaking, $a^n = a \times a \times ... \times a$ for n times. Sequentially, we can prove the following:

$$\begin{array}{lll} [a] & a^{m+n} = a^m \times a^n & \text{for } a \in \mathbb{R} \text{ and } m, n \in \mathbb{N} \\ [b] & a^{mn} = (a^m)^n & \text{for } a \in \mathbb{R} \text{ and } m, n \in \mathbb{N} \\ [c] & (ab)^n = a^n \times b^n & \text{for } a, b \in \mathbb{R} \text{ and } n \in \mathbb{N} \end{array}$$

Secondly, we define integer exponentiation. For $a \in \mathbb{R}_{\times}$, and $p \in \mathbb{Z}^+$,

$$\begin{aligned} a^0 &:= 1 \\ a^{-p} &:= \frac{1}{a^p} \end{aligned}$$

and similarly, formulae [a], [b], [c] still holds but 0 should be excluded from the values of those a's, and all $\mathbb{N}$ should be replaced by $\mathbb{Z}$.

Thirdly, we proceed to rational exponentiation. However, before defining it directly, there're some properties we're going to exhibit. The fact is that, for $a \in \mathbb{R}^+$, $p, q, r, s \in \mathbb{N}$, if $\frac{q}{p} = \frac{s}{r}$, then

$$\sqrt[p]{a^q} = \sqrt[p]{\sqrt[r]{(a^q)^r}} = \sqrt[pr]{a^{qr}} = \sqrt[pr]{a^{ps}} = \sqrt[r]{\sqrt[p]{(a^s)^p}} = \sqrt[p]{a^q}$$

With the promise of the property, the following definition is well-defined: Whenever a rational number s is expressed as a fraction $\frac{q}{p}$ with $q \in \mathbb{Z}$ and $p \in \mathbb{N}$, the expression $a^s = a^{\frac{q}{p}}$ is defined as

$$\sqrt[p]{a^q}$$

then laws of exponentiation similarly holds.

Something that is not so easy to deal with is the real exponentiation. The main idea follows from the fact that if $\{s_n\}_{n=1}^{\infty}$ is a monotone sequence of rational numbers and converges to $\alpha \in \mathbb{R}$, then the sequence $\{a^{s_n}\}_{n=1}^{\infty}$ converges, and moreover, if $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are two such increasing and decreasing sequences respectively, then $\{a^{s_n}\}_{n=1}^{\infty}$ and $\{a^{t_n}\}_{n=1}^{\infty}$ converge to the same number. This indicates that when $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are such monotone sequences, $\{a^{s_n}\}_{n=1}^{\infty}$ and $\{a^{t_n}\}_{n=1}^{\infty}$ converge to the same number.

For $a \in \mathbb{R}^+$, $r \in \mathbb{R}$, expressed as the limit of a monotone rational sequence $\{r_n\}$ (such one must exist since we can choose the digital expression),

$$a^r := \lim_{n\to\infty} a^{r_n}$$

Let $a \in \mathbb{R}^+$, and $\{r_n\}, \{s_n\}$ be rational sequences converging to $r, s \in \mathbb{R}$, respectively. Then $\{r_n + s_n\}_{n=1}^{\infty} \to r + s$, and

$$a^r a^s = \lim_{n\to\infty} a^{r_n} \lim_{n\to\infty} a^{s_n} = \lim_{n\to\infty} a^{r_n} a^{s_n} = \lim_{n\to\infty} a^{r_n+s_n} = a^{r+s}$$

Similarly we can prove the other laws.

Recall from De Moivre Theorem that for $a+bi \in \mathbb{C}$, expressed as $r(\cos\theta + i\sin\theta)$, and $n \in \mathbb{N}$, we have

$$(a+bi)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

and that for $z^n = r(\cos\theta + i\sin\theta)$, then we have

$$z = \sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}) \times \omega^k \qquad \text{where } k = 0, 1, 2, ..., n-1$$

Since we can find the n-th roots of a given complex number, we might try to define its exponentiation. However, this aspect has problems, which comes from the demand on "argument". I include the idea in the book on complex analysis written by Robert B Ash, and give the definition:

If $z = a + bi$, written in the form $r(\cos\theta + i\sin\theta)$, with $\theta \in [\alpha, \alpha + 2\pi)$, $Q = \frac{q}{p} \in \mathbb{Q}$, then we define the exponentiation "with respect to α", as follow

$$z_{(\alpha)}^{\frac{q}{p}} = \sqrt[\frac{q}{p}]{r}(\cos\frac{q}{p}\theta + i\sin\frac{q}{p}\theta)$$

Note that the "arguments" acts like the group $\mathbb{R}_{[\phi,\theta)}$, which is the continuity form with respect to the group $\mathbb{Z}_n$. For example, if we choose $\alpha = 0$, then the range of "argument" of a complex number must be in $[0, 2\pi)$, as we have learned in senior high. We can compare

$$\begin{aligned}
&(-1-\sqrt{3}i)_{(0)}^{\frac{5}{2}} \times (-1-\sqrt{3}i)_{(0)}^{\frac{1}{4}} \\
&= [2(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi)]_{(0)}^{\frac{5}{2}} \times [2(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi)]_{(0)}^{\frac{1}{4}} \\
&=_{\text{by definition}} 2^{\frac{5}{2}}(\cos(\frac{5}{2}\times\frac{4}{3}\pi) + i\sin(\frac{5}{2}\times\frac{4}{3}\pi) \times 2^{\frac{1}{4}}(\cos(\frac{1}{4}\times\frac{4}{3}\pi) + i\sin(\frac{1}{4}\times\frac{4}{3}\pi)) \\
&=_{\text{by properties}} 2^{\frac{11}{4}} \times (\cos(\frac{11}{4}\times\frac{4}{3}\pi) + i\sin(\frac{11}{4}\times\frac{4}{3}\pi)) \\
&= 2^{\frac{11}{4}} \times (\cos(\frac{5}{3}\pi) + i\sin(\frac{5}{3}\pi))
\end{aligned}$$

in operation in $\mathbb{C}$ and

$$\begin{aligned}
&\frac{5}{2}\times\frac{4}{3}\pi + \frac{1}{4}\times\frac{4}{3}\pi \\
&= \frac{4}{3}\pi + \frac{1}{3}\pi \\
&= \frac{5}{3}\pi
\end{aligned}$$

in operation in $\mathbb{R}_{[0,2\pi)}$, and find that the part of the "argument" in the former is (algetraic) homomorphic to the latter. This indicates, to observe the laws of exponentiation, it suffices to do it in the homomorphic structure $\mathbb{R}_{[\alpha,\alpha+2\pi)}$.

Hence we seems to have the following:
For any $z, w \in \mathbb{C}_\times$, $u, v \in \mathbb{Q}$, $\alpha \in \mathbb{R}$, we have the formulae

$$
\begin{aligned}
&[\alpha] \quad z_\alpha^{u+v} = z_\alpha^u \times z_\alpha^v \\
&[\beta] \quad z_\alpha^{uv} = (z_\alpha^u)_\alpha^v \\
&[\gamma] \quad (zw)_\alpha^u = z_\alpha^u \times w_\alpha^u
\end{aligned}
$$

Problem: Prove or disprove it.