# A problem

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[Problem] Let $A = \{\text{functions from } (0,1) \text{ to } \mathbb{R}\}$, and $B = \{\text{functions from } (0,1) \text{ continuously to } \mathbb{R}\}$. Show that (a) $A \not\approx \mathbb{R}$, and (b) $B \approx \mathbb{R}$.

*Proof.* We now show(a). Assume that $A \approx (0,1)$. Let $F : (0,1) \to A$ be such a bijection. Define a function $g : (0,1) \to \mathbb{R}$ such that for $c \in (0,1)$,

$$g(c) = \begin{cases} 0, & \text{if } (F(c))(c) = 1; \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

(This is the "diagonal argument" on continuum.) We know that $g \in A = F((0,1))$. For $x \in (0,1)$, since $g(x) \neq (F(x))(x)$, we have that $g \neq F(x)$. This means $g \notin F((0,1))$, a contradiction.

Therefore, $A \not\approx \mathbb{R}$.

Secondly, we prove (b). To describe a function from $(0,1)$ continuously to $\mathbb{R}$, we only need to know its values on rationals. Moreover, the set

$$C(\mathbb{Q} \cap (0,1), \mathbb{R}) \subset \mathbb{R}^{\mathbb{Q} \cap (0,1)}.$$

Note that restriction sets up a bijection between $B$ and $C(\mathbb{Q} \cap (0,1), \mathbb{R})$. So we have

$$|B| \leq |\mathbb{R}^{\mathbb{Q} \cap (0,1)}| = \mathfrak{c}.$$

We also know that $|B| \geq \mathfrak{c}$ because

$$B \supset \bigcup_{r \in \mathbb{R}} \{f : f(x) \equiv r\}.$$

Therefore, $|B| = \mathfrak{c}$, i.e. $B \approx \mathbb{R}$. □