Gamma Function and Balls In $\mathbb{R}^n$

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The Gamma function (a generalization of the factor function $n!$) is defined as $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ and the Beta funcion is $B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Then we have these properties:

(i) $\Gamma(x + 1) = x\Gamma(x)$.

(ii) $\Gamma(n + 1) = n!$.

(iii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(iv) $\Gamma(m + \frac{1}{2}) = \frac{(2m)(2m-1)\cdots(m+2)(m+1)}{2^{2m}}\sqrt{\pi}$.

(v) $B(x, y) = \int_0^1 (1 - z)^{x-1} z^{y-1} dz$.

To evidence (iv),

$$\begin{aligned}
\Gamma(m + \frac{1}{2}) &= (m - 1 + \frac{1}{2})(m - 2 + \frac{1}{2}) \cdots (0 + \frac{1}{2})\Gamma(\frac{1}{2}) \\
&= \frac{2m - 1}{2} \cdot \frac{2m - 3}{2} \cdots \frac{1}{2}\sqrt{\pi} \\
&= \frac{(2m)(2m - 1)(2m - 2) \cdots (4)(3)(2)(1)}{2^m \cdot 2^m \cdot (m)(m - 1)(m - 2) \cdots (2)(1)}\sqrt{\pi} \\
&= \frac{(2m)!}{2^{2m} \cdot m!} = \frac{(2m)(2m - 1) \cdots (m + 2)(m + 1)}{2^{2m}}\sqrt{\pi}.
\end{aligned}$$

Supposely we know that the volumn of a ball in $\mathbb{R}^3$ with radius r is $V_3(r) = \frac{4}{3}\pi r^3$. Find that in $\mathbb{R}^4$ and in $\mathbb{R}^5$, i.e. $V_4(r)$ and $V_5(r)$.

$$
\begin{aligned}
V_4(r) &= \int_{-r}^{r} V_3(\sqrt{r^2-x^2})dx = \int_{-r}^{r} \frac{4}{3}\pi(\sqrt{r^2-x^2})^3 dx \\
&= \frac{4}{3}\pi \int_{-r}^{r} (r-x)^{\frac{3}{2}}(r+x)^{\frac{3}{2}} dx =_{(x:=2ry \,\Rightarrow\, dx=2rdy)} \frac{4}{3}\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (r-2ry)^{\frac{3}{2}}(r+2ry)^{\frac{3}{2}} dy \\
&= \frac{4}{3}\pi(2r)^3(2r) \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{2}-y)^{\frac{3}{2}}(\frac{1}{2}+y)^{\frac{3}{2}} dy =_{(y:=z-\frac{1}{2} \,\Rightarrow\, dy=dz)} \frac{4}{3}\pi(2r)^4 \int_0^1 (1-z)^{\frac{3}{2}} z^{\frac{3}{2}} dz \\
&= \frac{4}{3}\pi(2r)^4 B(\frac{5}{2},\frac{5}{2}) = \frac{4}{3}\pi(2r)^4 \frac{(\Gamma(\frac{5}{2}))^2}{\Gamma(4+1)} = \frac{4}{3}\pi 16 r^4 \frac{(\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi})^2}{4!} \\
&= \frac{1}{2}\pi^2 r^4
\end{aligned}
$$

For $V_5(r)$,

$$
\begin{aligned}
V_5(r) &= \int_{-r}^{r} V_4(\sqrt{r^2-x^2})dx = \int_{-r}^{r} \frac{1}{2}\pi^2(\sqrt{r^2-x^2})^4 dx = \frac{1}{2}\pi^2 \int_{-r}^{r} (r^2-x^2)^2 dx \\
&= \frac{1}{2}\pi^2 \int_{-r}^{r} (r^4 - 2r^2x^2 + x^4) dx = \frac{1}{2}\pi^2 (r^4x - \frac{2}{3}r^2x^3 + \frac{1}{5}x^5)_{-r}^{r} = \frac{8}{15}\pi^2 r^5
\end{aligned}
$$

If we try $V_6(r)$, we'll find $V_6(r) = \frac{1}{6}\pi^3 r^6$. It seems that $V_n(r)$ is of the form $V_n(r) = a_n \pi^{k_n} r^n = C_n r^n$, where $\{k_n\}_{n=2}^{\infty} = \langle 1,1,2,2,3,3,4,4,\cdots\rangle$.

To reach a recurrance relation between $V_n(r)$ and $V_{n+1}(r)$, Assume $V_n(r) = C_n r^n$, we perform the following:

$$
\begin{aligned}
V_{n+1}(r) &= \int_{-r}^{r} C_n(\sqrt{r^2-x^2})^n dx \\
&= C_n \int_{-r}^{r} (r-x)^{\frac{n}{2}}(r+x)^{\frac{n}{2}} dx \\
&= (2r)^n C_n (2r) \int_{\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{2}-y)^{\frac{n}{2}}(\frac{1}{2}+y)^{\frac{n}{2}} dy \\
&= (2r)^{n+1} C_n \int_0^1 (1-z)^{\frac{n}{2}} z^{\frac{n}{2}} dz \\
&= (2r)^{n+1} C_n B(\frac{n}{2}+1, \frac{n}{2}+1) \\
&= (2r)^{n+1} C_n \frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(n+2)} = \frac{(2r)^{n+1}(\Gamma(\frac{n}{2}+1))^2}{(n+1)!} C_n \\
&= \frac{2^{n+1}C_n}{(n+1)!}(\Gamma(\frac{n}{2}+1))^2 r^{n+1}
\end{aligned}
$$

This means:

$$
C_{n+1} = \frac{2^{n+1}}{(n+1)!}(\Gamma(\frac{n}{2}+1))^2 C_n
$$

Let us try more cases. In fact, we have:

$$\begin{aligned}
C_2 &= \pi = \frac{1}{1!}\pi \\
C_3 &= \frac{4}{3}\pi = \frac{2^4}{3\cdot 4}\pi \\
C_4 &= \frac{1}{2}\pi^2 = \frac{1}{2!}\pi^2 \\
C_5 &= \frac{2^3}{3\cdot 5}\pi^2 = \frac{2^6}{4\cdot 5\cdot 6}\pi^2 \\
C_6 &= \frac{1}{3!}\pi^3 \\
C_7 &= \frac{2^5}{7\cdot 6\cdot 5}\pi^3 = \frac{2^8}{5\cdot 6\cdot 7\cdot 8}\pi^3 \\
C_8 &= \frac{1}{4!}\pi^4 \\
C_9 &= \frac{2^9}{5\cdot 6\cdot 7\cdot 8\cdot 9}\pi^4 = \frac{2^{10}}{6\cdot 7\cdot 8\cdot 9\cdot 10}\pi^4.
\end{aligned}$$

Observing more, we might find that

$$\begin{aligned}
C_{2j} &= \frac{\pi^j}{j!} = \frac{\pi^{\frac{2j}{2}}}{\Gamma(\frac{2j}{2}+1)}. \\
C_{2j+1} &= \frac{2^{2(j+1)}}{(j+1+1)(j+1+2)\cdots(j+1+j+1)\sqrt{\pi}}\pi^{j+\frac{1}{2}} \\
&= \frac{\pi^{\frac{2j+1}{2}}}{\Gamma(j+1+\frac{1}{2})} = \frac{\pi^{\frac{2j+1}{2}}}{\Gamma(\frac{2j+1}{2}+1)}.
\end{aligned}$$

Hence, for general case,

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}r^n$$

To show this, we know that when $n = 2$, the formula holds, so now if we also know that

$V_k(r) = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} r^k$, we hope to evalute V_{k+1}.

$$\begin{aligned}
V_{k+1}(r) &= \int_{-r}^{r} V_k(\sqrt{r^2 - x^2}) dx \\
&= \int_{-r}^{r} \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} (r^2 - x^2)^{\frac{k}{2}} dx \\
&= \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \int_{-r}^{r} (r-x)^{\frac{k}{2}} (r+x)^{\frac{k}{2}} dx \\
&=_{(x:=2ry \ \Rightarrow \ dx=2rdy)} \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} (r-2ry)^{\frac{k}{2}} (r+2ry)^{\frac{k}{2}} 2rdy \\
&=_{(Similarly)} \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} (2r)^{k+1} B(\frac{k}{2}+1, \frac{k}{2}+1) = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} (2r)^{k+1} \frac{(\Gamma(\frac{k}{2}+1))^2}{\Gamma(k+2)} \\
&= \frac{\pi^{\frac{k}{2}} (2r)^{k+1} \Gamma(\frac{k}{2}+1)}{(k+1)!}
\end{aligned}$$

If $k = 2p+1$, namely, odd, then

$$\begin{aligned}
(*) \ &= \frac{\pi^{\frac{k}{2}} r^{k+1} 2^{2p+2} \Gamma(p+1+\frac{1}{2})}{(2p+2)!} \\
&= \pi^{\frac{k}{2}} r^{k+1} \frac{2^{2p+2} (p+\frac{1}{2}) \cdots (0+\frac{1}{2}) \Gamma(\frac{1}{2})}{(2p+2)!} \\
&= \pi^{\frac{k}{2}} r^{k+1} \frac{2^{2p+2} \frac{2p+1}{2} \cdot \frac{2p-1}{2} \cdots \frac{1}{2} \Gamma(\frac{1}{2})}{(2p+2)!} \\
&= \pi^{\frac{k}{2}} r^{k+1} \frac{2^{2p+2} (\frac{1}{2})^{p+1} (2p+1) \cdots (1) \Gamma(\frac{1}{2}) (2p)(2p-2) \cdots (4)(2)}{(2p+2)! \cdot (2p)(2p-2) \cdots (4)(2)} \\
&= \frac{\pi^{\frac{k}{2}} r^{k+1} 2^{2p+2} (\frac{1}{2})^p (2p+1)! \sqrt{\pi}}{(2p+2)! 2^p p!} = \frac{\pi^{\frac{k+1}{2}} r^{k+1}}{(p+1)!} = \frac{\pi^{\frac{k+1}{2}} r^{k+1}}{\Gamma(\frac{k+1}{2}+1)}.
\end{aligned}$$

Else if $k = 2p$, even, then

$$\begin{aligned}
(*) \ &= \frac{\pi^{\frac{k}{2}} r^{k+1} 2^{2p+1} \Gamma(p+1)}{(2p+1)!} \\
&= \frac{\pi^{\frac{k}{2}} r^{k+1} 2^{2p+1} p!}{(2p+1)!} = \frac{\pi^{\frac{k}{2}} r^{k+1} 2^{p+1}}{\frac{(2p+1)(2p)(2p-1)\cdots(4)(3)(2)(1)}{(2p)(2p-2)\cdots(4)(2)}} \\
&= \frac{\pi^{\frac{k+1}{2}} r^{k+1}}{(p+\frac{1}{2})(p-1+\frac{1}{2}) \cdots (\frac{1}{2}) \sqrt{\pi}} = \frac{\pi^{\frac{k+1}{2}} r^{k+1}}{\Gamma(p+1+\frac{1}{2})} = \frac{\pi^{\frac{k+1}{2}} r^{k+1}}{\Gamma(\frac{k+1}{2}+1)}.
\end{aligned}$$

Mathematical induction then yields that $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.