Here I will give another proof concerning Abel's counterexample about a discontinuous function which is an infinite sum of continuous functions. The difference of this proof from the previous one shown in another article of mine is that I avoid the technique about the First Mean Value Theorem of Integration because it is not mentioned in standard high school courses.

The problem is that, we are asked to show that for $x \in (0, \pi]$,

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}. \tag{1}$$

We firstly consider the functions $\{F_N(x)\}_{N=1}^{\infty}$, defined by

$$F_N(x) := \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ... + \frac{\sin Nx}{N} \tag{2}$$

Then we have

$$\begin{aligned} F_N'(x) &= \cos x + \cos 2x + ... + \cos Nx \\ 2F_N'(x) \sin \frac{x}{2} &= \sin \frac{N+1}{2} x - \sin \frac{x}{2} \\ F_N'(x) &= \frac{1}{2} (\frac{\sin \frac{N+1}{2} x}{\sin \frac{x}{2}} - 1). \end{aligned} \tag{3}$$

We now get that

$$F_N(x) = -\frac{1}{2} \int_x^{\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt - \frac{x}{2} + C_N.$$

Choosing $x = \pi$, we find that $C_N = \frac{\pi}{2}$. For a fixed $x$, we find that

$$p\pi \leq \frac{N+1}{2} x < (p+1)\pi.$$

for some $p \in \mathbb{N} \cup \{0\}$. Hence

$$\begin{aligned} g_N(x) :&= \int_x^{\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt \\ &= \int_x^{\frac{2}{N+1}(p+1)\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt + \int_{\frac{2}{N+1}(p+1)\pi}^{\frac{2}{N+1}(p+2)\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt \\ &\quad + \int_{\frac{2}{N+1}(p+2)\pi}^{\frac{2}{N+1}(p+3)\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt + \cdots + \int_{\frac{2}{N+1}(p+q)\pi}^{\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt \end{aligned}$$

where $q \in \mathbb{N}$ is such that $\pi - \frac{2}{N+1}(p+q)\pi < \frac{2}{N+1}\pi$. Note that when $j = 1, 2, \cdots, q-1$,

$$g_{N,j} := \int_{\frac{2}{N+1}(p+j)\pi}^{\frac{2}{N+1}(p+j+1)\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt$$

is such that

$$
\begin{aligned}
0 \le g_{N,j} &\le \int_{\frac{2}{N+1}(p+j)\pi}^{\frac{2}{N+1}(p+j+1)\pi} \frac{\sin\frac{N+1}{2}t}{\sin\frac{x}{2}} dt \\
&= \frac{1}{\sin\frac{x}{2}} \left(\frac{-2}{N+1} \cos\frac{N+1}{2}t\right|_{\frac{2}{N+1}(p+j)\pi}^{\frac{2}{N+1}(p+j+1)\pi}) \\
&\le \frac{1}{\sin\frac{x}{2}} \frac{4}{N+1}
\end{aligned}
$$

whenever $p+j$ is even and while $p+j$ is odd,

$$
\begin{aligned}
0 \ge g_{N,j} &\ge \int_{\frac{2}{N+1}(p+j)\pi}^{\frac{2}{N+1}(p+j+1)\pi} \frac{\sin\frac{N+1}{2}t}{\sin\frac{x}{2}} dt \\
&= \frac{1}{\sin\frac{x}{2}} \left(\frac{-2}{N+1} \cos\frac{N+1}{2}t\right|_{\frac{2}{N+1}(p+j)\pi}^{\frac{2}{N+1}(p+j+1)\pi}) \\
&\ge \frac{1}{\sin\frac{x}{2}} \frac{-4}{N+1}
\end{aligned}
$$

This means, by squeeze theorem, that as $N \to \infty$, $g_{N,j} \to 0$ for all $j$'s. So

$$F_N(x) \to -\frac{x}{2} + \frac{\pi}{2}.$$

i.e.

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \lim_{n\to\infty} F_N(x) = \frac{\pi - x}{2}. \tag{4}$$