Complex Exponentiation

Definition Let *z* be a complex number, we know by Abel theorem that the series $\sum_{k=1}^{\infty}\frac{z^k}{k!}$ converges for any *z*. Define exp(*z*) = $\sum_{k=0}^{\infty}\frac{z^k}{k!}$.

Theorem Let *z*,*w* be complex numbers. Then exp(*z*+*w*) = exp(*z*) exp(*w*).

Proof Let $S_n = \left(1+\frac{z^1}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}\right)\left(1+\frac{w^1}{1!}+\frac{w^2}{2!}+\ldots+\frac{w^n}{n!}\right)$. We find by expanding that

$$S_n = \sum_{k=0}^{n}\frac{(z+w)^k}{k!}+T_n$$

where

$$T_n = \sum_{\mathrm{k=1}}^{\mathrm{n}}\left(\frac{\mathrm{z}^{\mathrm{k}}w^n}{k!n!}+\frac{z^{k+1}w^{n-1}}{(k+1)!(n-1)!}+\ldots+\frac{z^n w^k}{n!k!}\right)$$

For k=1,2,3,…,n, let $\rho = \max(|\mathrm{z}|,|\mathrm{w}|,1)$, we have

$$\left|\frac{\mathrm{z}^{\mathrm{k}}w^n}{k!n!}+\frac{z^{k+1}w^{n-1}}{(k+1)!(n-1)!}+\ldots+\frac{z^n w^k}{n!k!}\right| \le \rho^{2n}\left(\frac{1}{k!n!}+\frac{1}{(k+1)!(n-1)!}+\ldots+\frac{1}{n!k!}\right)$$

$$\le \rho^{2n}\cdot n\cdot\frac{1}{\left(\left[\frac{k+n}{2}\right]!\right)^2} \le \rho^{2n}\cdot n\cdot\frac{1}{\left(\left[\frac{n}{2}\right]!\right)^2}$$

So we have

$$|\mathrm{T_n}| \le \sum_{k=1}^{n}\rho^{2n}\cdot n\cdot\frac{1}{\left(\left[\frac{n}{2}\right]!\right)^2} \le 10\rho^{2n}\cdot\left[\frac{n}{2}\right]^2\cdot\frac{1}{\left(\left[\frac{n}{2}\right]!\right)^2}$$

$$= 10\rho^{2n}\cdot\left[\frac{n}{2}\right]^2\cdot\frac{1}{\left(\left[\frac{n}{2}\right]!\right)^2} \le 10\rho^{100\left(\left[\frac{n}{2}\right]-1\right)}\frac{1}{\left(\left(\left[\frac{n}{2}\right]\text{-}1\right)!\right)^2} \to 0$$

Hence $T_n \to 0$, $S_n \to \sum_{k=0}^{\infty}\frac{(z+w)^k}{k!} = \exp(\mathrm{z}+\mathrm{w})$. Since $S_n \to \exp(\mathrm{z})\exp(w)$ by definition, therefore the theorem is proved.

Theorem Let $\theta \in \mathbb{R}$. Then $\exp(i\theta) = \cos\theta + i\sin\theta$.

Proof Since

$$\exp(i\theta) = \sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!} = \sum_{j=0}^{\infty}\left(\frac{(i\theta)^{2j}}{(2j)!} + \frac{(i\theta)^{2j+1}}{(2j+1)!}\right) = \sum_{j=0}^{\infty}\left(\frac{(-1)^j\theta^{2j}}{(2j)!} + i\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right)$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j}}{(2j)!} + i\cdot\sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!} = \cos\theta + i\sin\theta$$

we are done.

Definition Let $\theta \in \mathbb{R}$. Define $cis\theta = \cos\theta + i\sin\theta$.

Theorem For $\alpha, \theta \in \mathbb{R}$, $e^{\alpha} cis\theta = \exp(\alpha + i\theta)$

Theorem The function *exp* is one to one from $\mathbb{R}^{+} \times [0,2\pi) \subseteq \mathbb{C}$ onto $\mathbb{C}_{\times}$.

Proof Since $\exp(\alpha + i\theta) = e^{\alpha} cis\theta = e^{\alpha}(\cos\theta + i\sin\theta)$ for α, θ in $\mathbb{R}^{+} \times [0,2\pi)$, hence it is one to one and onto.

Definition The inverse function of *exp* restricted on $\mathbb{R}^{+} \times [0,2\pi)$ is denoted by Log.

Definition For $Z \neq 0$, $W \in \mathbb{C}$. Define $Z^{W} = \exp(W\, \mathrm{Log}(Z))$.

Theorem Hence $Z^{W_1+W_2} = Z^{W_1}Z^{W_2}$, and $Z^{W_1W_2} = \left(Z^{W_1}\right)^{W_2}$.

Proof By definition, we have

$$Z^{W_1+W_2} = \exp((W_1 + W_2)Log(Z)) = \exp(W_1 Log(Z) + W_2 Log(Z))$$
$$= \exp(W_1 Log(Z))\exp(W_2 Log(Z)) = Z^{W_1}Z^{W_2}$$

and

$$Z^{W_1W_2} = \exp((W_1W_2)Log(Z)) = \exp(W_2 \mathrm{Log}(\exp(W_1 LogZ)))$$
$$= \left(\exp(W_1 LogZ)\right)^{W_2} = \left(Z^{W_1}\right)^{W_2}$$

Hence, these identities are proved.

Definition Define $\log_Z W = \dfrac{LogW}{LogZ}$.

Theorem For $Z \neq 0$, we have $LogZ = \ln|Z| + i\arg(Z)$.

Proof Let $Z = r(\cos\theta + i\sin\theta)$, where $\theta = \arg(Z)$. Then

$$LogZ = Log\left(r\left(\cos\theta + i\sin\theta\right)\right) = Log\left(Exp\left(\ln r\right)Exp\left(i\theta\right)\right)$$
$$= Log\left(Exp\left(\ln r + i\theta\right)\right) = \ln r + i\arg\theta = \ln|Z| + i\arg\theta$$

Theorem $Z^{\log_z W} = W$, $\log_Z Z^W = W$.

Proof We have

$$Z^{\log_z W} = Z^{\frac{LogW}{LogZ}} = Exp(\frac{LogW}{LogZ}\cdot LogZ) = ExpLogW = W$$

and

$$\log_Z Z^W = \frac{LogZ^W}{LogZ} = \frac{LogExp\left(W\cdot LogZ\right)}{LogZ} = \frac{W\cdot LogZ}{LogZ} = W.$$

Theorem When $W, Z\in R$, the new defined exponentiation Z^W coincides with the original definition. And then we find that the new form is indeed an extension of exponentiation.

Note: It suffices to show that the restruction of the new definitions (in complex systems) to real number system and the old definitions (in real systems as we learned in senior high) coincide.