This article would involve some thoughts about ways to solve certain basic differential equations. By a simple list I wwould write them down:

(1) $y' + p(x)y = q(x)$.

(2) $y" + (\alpha + \beta)y' + \alpha\beta \cdot y = 0$.

(3) $y" + 2r \cdot y' + r^2 y = 0$

(4) $y" + |a|y = 0$

(5) $y" + \{(\rho + i\omega) + (\rho - i\omega)\}y' + (\rho^2 + \omega^2)y = 0$.

They can all be solved throughout elementary calculus. However, we should be sensitive on the form, appearance of these equations.

The left hand side of the first one looks like the product rule — in fact, we can imagine that it is in the form

$$y'\varphi(x) + \varphi'(x)y$$

But our problem is: the coefficient of $y"$ is 1, how can there be a $\varphi(x)$ together with y' in above expression? This means that we must change the form of the equation. But why not multiply both sides by $\varphi(x)$?

$$y'\varphi(x) + \varphi(x)p(x)y = q(x)\varphi(x)$$

From this equation, we think that if $\varphi(x)$ is that function such that $\varphi(x)p(x) = \varphi'(x)$, then the problem is almost done. We hope that $\varphi(x)$ is not too bad — we want to see whether it can be differentiable and always positive. If

$$\varphi'(x) = \varphi(x)p(x)$$

i.e.

$$(log(\varphi(x)))' = \frac{\varphi'(x)}{\varphi(x)} = p(x)$$

then

$$log(\varphi(x)) = P(x) + C$$

where $P(x)$ is such that $P'(x) = p(x)$ and C is some real number. This means the solution must be in the form $\varphi(x) = C_0 \cdot e^{P(x)}$ for some C_0 and we may choose

$$\varphi(x) = e^{P(x)}$$

Back to the preceding step, for that φ, we find

$$(e^{P(x)}y)' = q(x)e^{P(x)}$$

and we get

$$e^{P(x)} \cdot y = R(x) + D$$

where $R(x)$ is such that $R'(x) = q(x)e^{P(x)}$, it might be $\int_0^x q(t)e^{P(t)}dt$ if $q(x)$ is not too bad. Finally,

$$y = e^{-P(x)} \cdot (R(x) + D)$$

Briefly speaking, if possible,

$$y = e^{-P(x)} \cdot (\int_0^x q(t)e^{P(t)}dt + D)$$

For the second type, we had better compare the form with quadratic polynomails. For $t^2 + (\alpha + \beta)t + \alpha\beta$, when performing factorization we find

$$\begin{aligned} t^2 + (\alpha + \beta)t + \alpha\beta &= t^2 + \alpha t + \beta t + \alpha\beta \\ &= t \cdot (t + \alpha) + \beta \cdot (t + \alpha). \end{aligned}$$

Applying this technique to differential operators we get

$$\begin{aligned} y" + (\alpha + \beta)y' + \alpha\beta \cdot y &= y" + \alpha y' + \beta y' + \alpha\beta \\ &= (y' + \alpha y)' + \beta(y' + \alpha y) \end{aligned} \tag{1}$$

Substituting Y_1 for $y' + \alpha y$, we get

$$Y_1 = y' + \alpha y = c_2 e^{-\beta x}$$

Similarly, Y_2 turns out to be

$$Y_2 := y' + \beta y = c_1 e^{-\alpha x}$$

Viewing y' and y as the variables in a system of simultaneous equations, we find that y is in the form

$$y = \widetilde{c}_1 e^{-\alpha x} + \widetilde{c}_2 e^{-\beta x}$$

Next, we proceed with type (3). The form

$$(y' + ry)' + r(y' + ry) = 0$$

indicates that $y' + ry = c_1 e^{-rx}$ for some c_1. Choosing $\varphi(x) = e^{rx}$, then type (1) yields that

$$\begin{aligned} (\varphi(x)y)' &= c_1 \\ \varphi(x)y &= c_1 x + c_0 \\ y &= c_0 e^{-rx} + c_1 x e^{-rx} \end{aligned}$$

which is the answer.

Then we now focus on second ordered equations with real number coefficients.