Addition Table

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According to *Edmund Landau's Foundations of Analysis* we construct the natural numbers $\mathbb{N}$ and its operations. Now an extra element 0 will add to the system to eliminate some problems in digit arithmetic.

0 is an element so that 0 and all members of $\mathbb{N}$ form the whole number set $\mathbb{W}$. The operations concerning 0 are such that

$$0 + k = k + 0 = k$$
$$0 \cdot k = k \cdot 0 = 0.$$

There is an specified element: $\mathbf{T} = 9'$.

Let $a : \mathbb{W}_{\leq n} \to \mathbb{W}$ be a function. We inductively define $\sum a$. Choose $s : \mathbb{W}_{\leq n} \to \mathbb{W}$ defined by $s(0) = a(0)$, $s(k) = a(k-1) + s(k)$. Then $\sum a := s(n)$. For convenience, we used to write $\sum_{k=0}^{n} a_k$ instead of $\sum a$.

Assume the *Division Algorithm*. It follows that for each $N \in \mathbb{W}$, there is a unique n and a unique sequence b such that $b : \mathbb{W}_{\leq n} \to \mathbb{W}$ and

$$N = \sum_{k=0}^{n} b_k \cdot \mathbf{T}^n.$$

At this time, we write $N = \bar{b}$, or in a familiar form, $\overline{b_n b_{n-1} \cdots b_2 b_1 b_0}$. The "bar" is usually omitted.

Hence $9' = \mathbf{T} = 1 \cdot \mathbf{T} + 0 = 10$, $10' = (1 \cdot \mathbf{T} + 0) + 1 = \mathbf{T} + (0 + 1) = 1 \cdot \mathbf{T} + 1 = 11$, and similarly, $11' = 12$, $12' = 13$, $13' = 14$, $14' = 15$, $15' = 16$, $16' = 17$, and $17' = 18$.

Now we show two lemmas.

Lemma.

$1+1=2$	$1+2=3$	$1+3=4$	$1+4=5$	$1+5=6$
$2+1=3$	$2+2=4$	$2+3=5$	$2+4=6$	$2+5=7$
$3+1=4$	$3+2=5$	$3+3=6$	$3+4=7$	$3+5=8$
$4+1=5$	$4+2=6$	$4+3=7$	$4+4=8$	$4+5=9$
$5+1=6$	$5+2=7$	$5+3=8$	$5+4=9$	$5+5=10$
$6+1=7$	$6+2=8$	$6+3=9$	$6+4=10$	$6+5=11$
$7+1=8$	$7+2=9$	$7+3=10$	$7+4=11$	$7+5=12$
$8+1=9$	$8+2=10$	$8+3=11$	$8+4=12$	$8+5=13$
$9+1=10$	$9+2=11$	$9+3=12$	$9+4=13$	$9+5=14$

$1+6=7$	$1+7=8$	$1+8=9$	$1+9=10$
$2+6=8$	$2+7=9$	$2+8=10$	$2+9=11$
$3+6=9$	$3+7=10$	$3+8=11$	$3+9=12$
$4+6=10$	$4+7=11$	$4+8=12$	$4+9=13$
$5+6=11$	$5+7=12$	$5+8=13$	$5+9=14$
$6+6=12$	$6+7=13$	$6+8=14$	$6+9=15$
$7+6=13$	$7+7=14$	$7+8=15$	$7+9=16$
$8+6=14$	$8+7=15$	$8+8=16$	$8+9=17$
$9+6=15$	$9+7=16$	$9+8=17$	$9+9=18$

Proof. By definition quite a few times, $1+1=1'=2$. Similarly, $2+1=2'=3$, $3+1=3'=4$, $4+1=4'=5$, $5+1=5'=6$, $6+1=6'=7$, $7+1=7'=8$, $8+1=8'=9$, and $9+1=9'=10$. Next, $1+2=1+1'=(1+1)'=2'=3$, $2+2=2+1'=(2+1)'=3'=4$, $3+2=3+1'=(3+1)'=4'=5$, $4+2=4+1'=(4+1)'=5'=6$, $5+2=5+1'=(5+1)'=6'=7$, $6+2=6+1'=(6+1)'=7'=8$, $7+2=7+1'=(7+1)'=8'=9$, $8+2=8+1'=(8+1)'=9'=10$, and $9+2=9+1'=(9+1)'=10'=11$.

Let's go on. $1+3=1+2'=(1+2)'=3'=4$, $2+3=2+2'=(2+2)'=4'=5$, $3+3=3+2'=(3+2)'=5'=6$, $4+3=4+2'=(4+2)'=6'=7$, $5+3=5+2'=(5+2)'=7'=8$, $6+3=6+2'=(6+2)'=8'=9$, $7+3=7+2'=(7+2)'=9'=10$, $8+3=8+2'=(8+2)'=10'=11$, $9+3=9+2'=(9+2)'=11'=12$. $1+4=1+3'=(1+3)'=4'=5$, $2+4=2+3'=(2+3)'=5'=6$, $3+4=3+3'=(3+3)'=6'=7$, $4+4=4+3'=(4+3)'=7'=8$, $5+4=5+3'=(5+3)'=8'=9$, $6+4=6+3'=(6+3)'=9'=10$, $7+4=7+3'=(7+3)'=10'=11$, $8+4=8+3'=(8+3)'=11'=12$, $9+4=9+3'=(9+3)'=12'=13$, $1+5=1+4'=(1+4)'=5'=6$, $2+5=2+4'=(2+4)'=6'=7$, $3+5=3+4'=(3+4)'=7'=8$, $4+5=4+4'=(4+4)'=8'=9$, $5+5=5+4'=(5+4)'=9'=10$, $6+5=6+4'=(6+4)'=10'=11$, $7+5=7+4'=(7+4)'=11'=12$, $8+5=8+4'=(8+4)'=12'=13$, $9+5=9+4'=(9+4)'=13'=14$, $1+6=1+5'=(1+5)'=6'=7$, $2+6=2+5'=(2+5)'=7'=8$, $3+6=3+5'=(3+5)'=8'=9$, $4+6=4+5'=(4+5)'=9'=10$, $5+6=5+5'=(5+5)'=10'=11$, $6+6=6+5'=(6+5)'=11'=12$, $7+6=7+5'=(7+5)'=12'=13$, $8+6=8+5'=(8+5)'=13'=14$, $9+6=9+5'=(9+5)'=14'=15$.

Moreover, $1+7=1+6'=(1+6)'=7'=8$, $2+7=2+6'=(2+6)'=8'=9$, $3+7=3+6'=(3+6)'=9'=10$, $4+7=4+6'=(4+6)'=10'=11$, $5+7=5+6'=(5+6)'=11'=12$, $6+7=6+6'=(6+6)'=12'=13$, $7+7=7+6'=(7+6)'=13'=14$, $8+7=8+6'=(8+6)'=14'=15$, $9+7=9+6'=(9+6)'=15'=16$, $1+8=1+7'=(1+7)'=8'=9$, $2+8=2+7'=(2+7)'=9'=10$, $3+8=3+7'=(3+7)'=10'=11$, $4+8=4+7'=(4+7)'=11'=12$, $5+8=5+7'=(5+7)'=12'=13$, $6+8=6+7'=(6+7)'=13'=14$, $7+8=7+7'=(7+7)'=14'=15$, $8+8=8+7'=(8+7)'=15'=16$, $9+8=9+7'=(9+7)'=16'=17$.

Finally, $1+9=1+8'=(1+8)'=9'=10$, $2+9=2+8'=(2+8)'=10'=11$, $3+9=3+8'=(3+8)'=11'=12$, $4+9=4+8'=(4+8)'=12'=13$, $5+9=5+8'=(5+8)'=13'=14$, $6+9=6+8'=(6+8)'=14'=15$, $7+9=7+8'=(7+8)'=15'=16$, $8+9=8+8'=(8+8)'=16'=17$, $9+9=9+8'=(9+8)'=17'=18$,

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