Here I write down an example about the fact that a pointwise convergent series of continuous functions does not necessarily converge to a continuous function. This example is derived by Abel: While $0<x \leq \pi$, we always have

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}. \tag{1}$$

To reach a proof, we consider the functions $\{F_N(x)\}_{N=1}^{\infty}$, defined by

$$F_N(x) := \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ... + \frac{\sin Nx}{N} \tag{2}$$

Then we have

$$\begin{aligned} F_N'(x) &= \cos x + \cos 2x + ... + \cos Nx \\ 2F_N'(x) \sin \frac{x}{2} &= \sin \frac{N+1}{2} x - \sin \frac{x}{2} \\ F_N'(x) &= \frac{1}{2} \left(\frac{\sin \frac{N+1}{2} x}{\sin \frac{x}{2}} - 1 \right). \end{aligned} \tag{3}$$

We now get that

$$\begin{aligned} F_N(x) &= -\frac{1}{2} \int_x^{\pi} \frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} dt - \frac{x}{2} + C_N = -\frac{1}{2} \left(\frac{1}{\sin \frac{x}{2}} \int_x^{\xi} \sin \frac{N+1}{2} t \right) - \frac{x}{2} + C_N \\ &= -\frac{1}{N+1} \frac{1}{\sin \frac{x}{2}} [\cos \frac{N+1}{2} t|_x^{\xi}] - \frac{x}{2} + C_N = \frac{1}{N+1} T(x) - \frac{x}{2} + C_N \end{aligned} \tag{4}$$

for some constant C_N, and some function $T(x)$. To compute C_N, we take x as $\frac{\pi}{2}$, and find that the LHS becomes $\sum_{j=0}^{N} \frac{(-1)^j}{2j+1}$, while the RHS is $\frac{1}{N+1} T(\frac{\pi}{2}) - \frac{\pi}{4} + C_N$. Therefore,

$$C_N = \sum_{j=0}^{N} \frac{(-1)^j}{2j+1} + \frac{\pi}{4} - \frac{1}{N+1} T(\frac{\pi}{2}) \tag{5}$$

By (4), (5), let N go to infinity, we find that $\{F_N(x)\}$ converges to

$$F(x) := \frac{\pi - x}{2} \tag{6}$$

This ends our partial discussion. We now have only to consider points other than this interval, then the "counterexample" is finished. In fact, This goal can be immediately done by observation on period of F(x). Since (6) also holds on $(0, 2\pi)$, we conclude that values of $F(x)$ on the whole real line are copies of values on $(0, 2\pi)$, and all those $2k\pi$'s will be sent to 0. .