A Quite Note On Set Theory

1 Sets

1.1 Basic Sets

Collected objects are often what mathematics focus on, so we have the notion of sets. A collection of objects "is called" a **set**. We care about the members of sets. Sets A and B "are called" **equal** if and only if they contain the same elements, by which I mean,

whenever $x \in A$, we have $x \in B$, and
whenever $y \in B$, we have $y \in A$.

1.2 Operations

Having sets, we hope to define operations on them.

DEFINITION. For sets A, B,
(i) $A \cap B := \{x \,|\, x \in A \text{ and } x \in B\}$;
(ii) $A \cup B := \{x \,|\, x \in A \text{ or } x \in B\}$;
(iii) $A \setminus B := \{x \,|\, x \in A \text{ but } x \notin B\}$.

Some properties are immediately.

PROPOSITION. Let S, T, U be sets.
(a) $(S \cap T) \cap U = S \cap (T \cap U)$.
(b) $(S \cup T) \cup U = S \cup (T \cup U)$.
(c) $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$.

PROOF. I only show (c). Let $x \in S \cup (T \cap U)$. Then $x \in S$ or $x \in T \cap U$. Our goal is to prove that $x \in S \cup T$ and $x \in S \cup U$. If $x \in S$, then $x \in S \cup T$ by definition of union. If $x \in T \cap U$, then $x \in T$ (definition of intersection). So $x \in S \cup T$ by definition of union.

Let $y \in (S \cup T) \cap (S \cup U)$. Then $y \in S \cup T$ and $y \in S \cup U$. We want to show that $y \in S$ or $y \in T \cap U$. If $y \in S$, then we finish the proof. Suppose that $y \notin S$. We want to show $y \in T$ and $y \in U$.

For $y \in T$, since $y \in S \cup T$ but $y \notin S$, by definition of union, we get that $y \in T$. Similarly, $y \in U$. Thus, by convention of set equality, we've shown the identity. □

1.3 Complements, Power Sets

Let A be a set. We call S a **subset** set of A if for any $x \in S$, it holds that $x \in A$.

Suppose that a "universal set" V is given. Let $A \subset V$. The **complement** of A with respect to V, denoted by A^{cV}, V always omitted, is defined as $A^c = V \setminus A$.

EXAMPLE. (a)If $V = \mathbb{R}$ is chosen, then $(\mathbb{R}^+)^c = \mathbb{R}_{\leq 0}$. (b)Also note that, $(A \cup B)^c = A^c \cap B^c$.

The **power set** of A is defined as

$$\wp(A) = \{S \,|\, S \subset A\}.$$

1.4 Ordered Pairs and Products

The ordered pair is another concept different from that of a set. We denote an **ordered pair** by (a, b). To distinguish an ordered pair from a set, note that ordered pairs have the property that

$$\text{If } (a, b) = (c, d) \text{ then } a = c\ ,\ b = d.$$

Similarly we also have the form $(a_1, \cdots, a_k)$ which contains k components. We omit the detail.

The purpose that we define an ordered pair is that we have the notion of a product set. The **product set** of two sets A and B is defined as

$$A \times B = \{(a,b) \,|\, a \in A \ , \ b \in B\}.$$

Also note that we immediately have the following properties.

(a) $A \times (B \cup C) = A \times B \cup A \times C$.

(b) $A \times (B \cap C) = A \times B \cap A \times C$.

We might as well convent the *priority* of operations of sets. I gave the order as: complement/set-minus, power set, product, intersection, union.

1.5 Index Operations

We have the thoughts about index sets. If we list the sets

$$A_1, A_2, A_3, \cdots ,$$

then the set $A = \cup_{k=1}^{\infty} A_k$ is given in the way that $x \in A$ if and only if $A \in A_k$ for some $k \in \mathbb{N}$. Similarly, $B = \cap_{k=1}^{\infty} A_k$ is given by $B = \{x \,|\, x \in A_k \text{ for any } k \in \mathbb{N}\}$.

Let $\mathscr{A}$ be a nonempty set of sets. Then the intersection and union are similarly defined. Namely,

$$\bigcap \mathscr{A} := \{x \,|\, x \in A \text{ for any } A \in \mathscr{A}\},$$
$$\bigcup \mathscr{A} := \{x \,|\, x \in A \text{ for some } A \in \mathscr{A}\}.$$

Exercise Knowing the limits, if $a_n > 0$ is a strictly decreasing sequence, show that $\cap_{k=1}^{\infty}[0, a_k) = [0, \lim_{k\to\infty} a_k]$.

2 Relations

2.1 Basic Form

A **relation** between a set A and a set B is a subset R of $\wp(A \times B)$. By the notion of relation we hope to classify or build relationship between given sets.

For a relation R between A and B, we want to know who is related to other and who other is related to. This is a definition.

Definition. On a relation R, the **domain**, and the **range**, of R are defined as: $dom(R) = \{x \in A \,|\, (x,y) \in f \text{ for some } y \in B\}$ and $ran(R) = \{y \in B \,|\, (x,y) \in R \text{ for some } x \in A\}$.

We also care about those relations with some specified properties.

Let R be a relation on A (i.e. a relation between A and itself).

(R) Reflexivity: If $(x,x) \in R$ for any $x \in A$, then we say R is reflexive.

(S) Symmetry: Suppose that if $(x,y) \in R$ then $(y,x) \in R$. Then we say R is symmectric.

(A) Anty-Symmetry: Suppose that if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$. Then we say R is anty-symmectric.

(T) Transitivity: Suppose that if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. Then we say R is transitive.

Example. $R_1 := \{(x,y) \in \mathbb{N} \times \mathbb{N} \,|\, x \leq y\}$, $R_2 := \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \,|\, m \equiv n \,(\text{mod }) 5\}$, $R_3 := \{(p,q) \in \mathbb{N} \times \mathbb{N} \,|\, p \text{ divides } q\}$, and $R_4 = \{(S,T) \,|\, S \subset T,\ S,T \subset \mathbb{R}\}$ are examples of each type above.

For writing and reading convenience, we always write

$$xRy$$

instead of $(x,y) \in R$.

2.2 Equivalence Relations

An **equivalence relation** is a reflexive, symmetric, transitive relation on some A. It is usually denoted by the symbol $\sim$. To be explicit, $\sim$ is an equivalence relation means: For each $x, y, z \in A$,

(i) $x \sim x$.

(ii) if $x \sim y$, then $y \sim x$.

(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.

EXAMPLE. $\sim_1 := R_3 = \{(p,q) \in \mathbb{N} \times \mathbb{N} \,|\, p \text{ divides } q\}$ is an equivalence relation on $\mathbb{Z}$.

EXAMPLE. $\sim_2 := \{(\langle x,y\rangle, \langle z,w\rangle) \in (\mathbb{N}\times\mathbb{N}) \times (\mathbb{N}\times\mathbb{N}) \,|\, xw = yz\}$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}$. Moreover, we observe that $\langle 1,2\rangle \sim \langle 2,4\rangle \sim \langle 3,6\rangle$, and $\langle 2,5\rangle \sim \langle 4,10\rangle \sim \langle 100,250\rangle$, which presents the notion of the rationals.

2.3 Equivalence classes

However, an important further observation is the classification on those elements in A by equivalence.

We make the definitions.

For an equivalence relation $\sim$ on a set A, given $x \in A$, we denote the equivalence class by $[x]_\sim$, sometimes omitting the index $\sim$, which is defined as

$$[x]_\sim = \{y \in A \,|\, y \sim x\}.$$

and the set of all equivalence classes of $\sim$ on A is called the **quotient set** of $\sim$ under A, denoted by $A/\sim$.

EXAMPLE. On $\mathbb{Z}$, using $\sim_3 := \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \,|\, 3|x-y\}$, which can be shown to be an equivalence relation. The equivalence classes are:

$$\begin{aligned} 3\mathbb{Z} &= \{\cdots, -3, 0, 3, 6, 9, \cdots\} \\ 3\mathbb{Z}+1 &= \{\cdots, -2, 1, 4, 7, 10, \cdots\} \\ 3\mathbb{Z}+2 &= \{\cdots, -4, -1, 2, 5, 8, \cdots\}. \end{aligned}$$

Thus, $\mathbb{Z}/\sim_3 = \{3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2\}$.

2.4 Partitions

This leads to the notion of a partition P on the given set A. Let $P = \{P_i\}_{i\in\mathscr{A}}$, where $P_i \subset A$ for all i. P is called a **partition** of A if

(1) all sets in P are *pairwisely disjoint* and
(2) $A = \bigcup_{P_i \in P} P_i$.

PROPOSITION. (1) If $\sim$ is an equivalence relation on A, then $A/\sim$ is a partition on A.

(2) If P is a partition on A, then

$$\begin{aligned} \sim_P : &= \{(x,y) \,|\, x, y \in P_i \text{ for some } P_i\} \\ &= \{(x,y) \,|\, x, y \text{ are in the same } P_i \\ &\quad \text{for some } P_i\} \end{aligned}$$

is an equivalence relation.

3 Functions

3.1 Definition

A function is a relation which has a "unique correspondence". Recall that, in the sense of greatnness, 7 is related to 6,5,4,3. However, functions do not allow such a case. We mean that, functions is retricted to avoid multiple correspondences.

DEFINITION. A function f from A to B is a relation between A to B such that (i)whenever $(x,y) \in f$ and $(y,z) \in f$, it holds that $y = z$. (ii) f satisfies $dom(f) = A$ and $ran(f) \subset B$.

There're some customary types for letting a function. The following are examples.

ORIGINAL TYPE: $f_1 := \{(x, 2x) \,|\, x \in \mathbb{R}^+\}$.
PREDICATE TYPE: $f_2(x) = 2x$, for $x \in \mathbb{R}^+$.
ASSIGNMENT TYPE: $f_3 : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto 2x$.

PROPOSITION. Two functions are equal if and only if they have the same domain and their assignments are equal for each element of the common domain.

3.2 One-to-one and Onto

There are some special types of functions.
DEFINITIONS.

(a) f is one-to-one if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

(b) f is onto if for $y \in B$, there is an $x \in A$ such that $f(x) = y$.

Definition. If f is one-to-one and onto, then f is called bijective, a bijection, or an one-one correspondence.
Example. (a) The function $g_1 : A \to A/\sim$, $x \mapsto [x]_\sim$ is onto. (b) The function $g_2 : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto 2x$ is an one-one correspondence.

3.3 Composition and Inverse

For two functions $f : A \to B, g : B' \subset B \to C$, we define their composition $g \circ f : A \to C$ by

$$x \mapsto g(f(x)).$$

Note that we have the associative law for function compositions.
Proposition. If $f : A \to B, g : B' \subset B \to C, h : C' \subset C \to D$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.
Example. Familiar cases are in this form: Let $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$. Then $g \circ f(x) = \frac{1}{x^2+1}$.

In the sense of finding a solution of a certain function (for example, showing a function onto), we need what is called the inverse of this function. We now give the definition in the viewpoint of its essence of being a relation.

Definition. Let f be a function. If $\grave{f}$ is also a function, then f is called invertable.

The main properties are:

Proposition. (a) f is invertable if and only if f is one-to-one and onto,

(b) which holds if and only if there is another function g such that $g(f(x)) = f(g(x)) = x$ for all proper x (i.e. all x such the terms above are defined). (Require the AC)

3.4 The Index

It's time to illustrate the index-sets. Since the sense of an index set is quite intuitive, yet I'll still emphasize the essence.

Let $\mathcal{I}$ be a set. an index set is a function $\mathscr{A} : \mathcal{I} \to \mathscr{M}$, where the set $\mathscr{M}$ is a much larger collection of sets. Note that its image (the same meaning as range)

$$ran(\mathscr{A}) = \{\mathscr{A}_i\}_{i \in \mathcal{I}}$$

is what we used to call an index set.

For example, if we're going to consider the sequence $\{[0, \frac{1}{n})\}_{n=1}^{\infty}$ of intervals in $\mathbb{R}$, it is, in fact, the **image** of the function $\mathscr{A} : \mathbb{N} \to \wp\mathbb{R}$ such that

$$n \to [0, \frac{1}{n})$$

4 Cardinality

4.1 Equinumerosity

A natural thought about "numbers" comes from certain classificatiion of collections of objects. We have the intuition in mind, that the string "1572xc" has 6 characters. Experience tells us that counting is a correspondence between the observing set and a standard "base" set.

To be more precisely, according to my intuition, to say $\{\mathbb{N}, (2,3), 1, 7, \frac{1}{\sqrt{10}}\}$ has 5 elements, I need to make an one-to-one correspondence between $\{1, 2, 3, 4, 5\}$ and $\{\mathbb{N}, (2,3), 1, 7, \frac{1}{\sqrt{10}}\}$, by which I mean

$$\begin{aligned} 1 &\leftrightarrow \mathbb{N}, \\ 2 &\leftrightarrow (2,3), \\ 3 &\leftrightarrow 1, \\ 4 &\leftrightarrow 7, \\ 5 &\leftrightarrow \frac{1}{\sqrt{10}}. \end{aligned}$$

This forms a function (bijection).

The first notion of counting is equinumerosity.

Definition. The sets A and B are called equinumerosity, if there is an one-to-one correspondence between A and B, and it's notation is $A \approx B$.

4.2 The set of all sets

Proposition. Let $\mathscr{A}$ be *a collection of sets.* Write

$$\sim := \{(A, B) \in \mathscr{A} \times \mathscr{A} \mid A \approx B\}.$$

Then $\sim$ is an equivalence relation on $\mathscr{A}$.

A **question** is that: Why don't we let A to be the collection of all sets? The answer is that: such a set does not exist.

(The fact is that, the proposition still holds by the same argument even if we change the words "a collection of sets" to the words "the set of all sets", but this statement is nonsense.)

Assume that we have a set B which contains all sets. Then we construct another set

$$Q := \{x \in B \mid x \notin x\}.$$

If $Q \in Q$, then $Q \in B$ and $Q \notin Q$, a contradiction. If $Q \notin Q$, since $Q \in B$ (definition of B), it holds that $Q \in Q$, a contradiction. Now, neither $Q \in Q$ nor $Q \notin Q$ holds, which is another contradiction. Hence such a B fails to exist. $\square$

Definition. If there is a function f from A one-to-one to B, then we say that B dominates A, or A is dominated by B, written $A \precsim B$.

4.3 The Integers

We have a series of equinumerosities.

Example.

(a) $\mathbb{N} \approx \mathcal{O}_{\mathbb{N}} := \{2n-1 \mid n \in \mathbb{N}\}$.

(b) $\mathbb{N} \approx \mathbb{W} := \{m \in \mathbb{Z} \mid m \geq 0\}$.

(c) $\mathbb{N} \approx \mathbb{Z}$.

Proof. The key way of showing that a set S is equinumerous to $\mathbb{N}$ is that, to list the set S unrepeatedly as a sequence. For (a), the sequence is $\{2, 4, 6, 8, ...\}$; for (b), we write $\{0, 1, 2, 3, 4, ...\}$; for (c), the sequence is $\{0, 1, -1, 2, -2, 3, -3, ...\}$. However, a precise proof is left. $\square$

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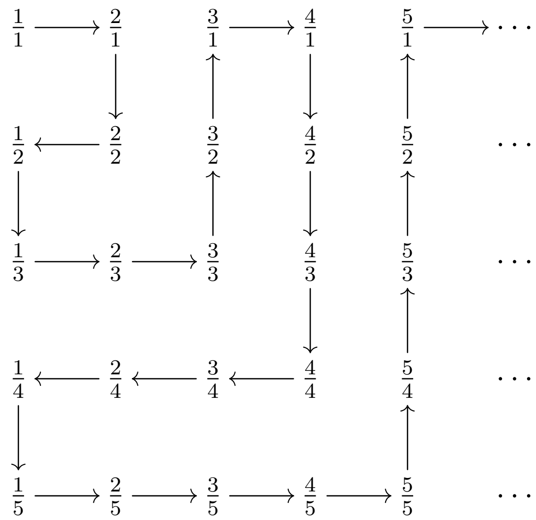
4.4 The Rationals

Example. $\mathbb{N} \approx \mathbb{Q}$.

An intuitive argument is, the following illustrating graph.



Note that each fraction is on the given path. I mean, if we follow this path, we can run over all fractions. The sequence that we're seeking comes from this path. When walking on this path, Since $\frac{2}{2} = \frac{1}{1}$, which is repeated, so we jump over it; $\frac{3}{3} = \frac{1}{1}$, $\frac{4}{2} = \frac{2}{1}$,..., which are all jumped over. Then, the resulting sequence is

$$\left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}, \frac{4}{1}, \frac{4}{3}, \frac{3}{4}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \cdots\right\}.$$

A precise proof is quite complicated, for it has many problems to conquer .

4.5 The Reals

Example. $\mathbb{Q} \prec \mathbb{R}$.

Note that this incredible result says that we can not write down the whole real number set in a sequence. This means, in any sequence, no matter how carefully we do, there are some real numbers that we fail to catch into this sequence.

At this time, I only mention the name of the method of the argument of the property. Cantor's **diagonal method** points out the main idea. With this trick, we can find the "losing element" mentioned in the last paragraph.

Note that during this long discussion, we will have proved the following statements.

(a) A subset of any "countable" set is "countable".

(b) $[0,1] \approx (0,1) \approx \mathbb{R}$.

4.6 The Complex

EXAMPLE. $\mathbb{R} \approx \mathbb{C}$.

When it comes to equinumerosity of $\mathbb{R}$ and $\mathbb{C}$, we ought to introduce Schröder-Berstein Theorem.

SCHRÖDER-BERSTEIN THEOREM. Let $f : A \to B$ and $g : B \to A$ be both one-to-one functions. Then there must be some one-to-one function h from A onto B.

Note that if we can find f and g from $\mathbb{R}$ to $\mathbb{C}$ and from $\mathbb{C}$ to $\mathbb{R}$ which are one-to-one, the proof is over. f is chosen as an embedding, while g is a rearrangement of decimal places of both real and imaginary part. The detail is omitted.

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4.7 Set Levels

DEFINITION. (a) A finite set is a set that is equinumerous to $\{1, 2, \cdots, n\}$ for some n, or it is empty. (b) A denumerable set is a set that is equinumerous to $\mathbb{N}$. A countable set is a set that is either finite or denumerable.

5 Cardinal Numbers

The concept of Cardinal Numbers comes from the thought that we want to give a symbol which presents how "many" elements are there in this set. So we have an assignment that send each set A to symbol, which we denote by $|A|$. Moreover, we have to be sensitive that this assignment has a request that

REQUEST. Let A, B be sets. Then $|A| = |B|$ if and only if $A \approx B$.

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So what is a cardinal number? The conclusion that we make now is an assigned symbol.

6 Ordered Sets

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7 Axiomization

A precise development of Set Theory requires some axioms, which can be viewed as a start of the theory. Every proposition is duducted from either axioms or from lower-leveled propositions.

A question is that, why to axiomize the theory?

We always "let a set", and do something on it without verifying its existence. This would be a danger. For example.

(1) Let $S := \{x : x \notin x\}$.

(2) Let $r = \sum_{k=0}^{\infty} 2^k$.

So we ought to quote some basic, intuitive, and reasonable statements as what are called axioms.

The reason by which we choose these statements as axioms is because that they seems to be required and will not self-contradict (i.e.

they would "never" logically lead to a contradiction). However, whether the seeming is real is a study in mathematical logic.

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From now on a new term "The cardinal number of A" will be added to undefined terms. The flowing axiom is used to start the discuss of cardinal number.

> Let A and B be sets. Then
> $A \approx B$ if and only if $|A| = |B|$.

Next I write down an equivalent statement of Axiom of Choice.

ZORN'S LEMMA. Let $(S, \preceq)$ be a partial order. Assume that every chain C of S has an upper bound in S, then S obtains a maximal element (i.e. there is no elements y in S such that $x \prec y$ where x is this maximal element).

In axiomatic (precise) developments of set theory, we prefer to defining a function (and hence a relation) rather than these between given sets A and B. A concrete distinguish is the case that we might require the assignment $x \mapsto \wp x$. No matter x is retricted in any S, a given set, we cannot directly write down the **codomain**

Note that the word is just introducted for convenience. It does not make too much difference in the theory. If $f : A \to B$, then we might say that B is the chosen codomain of f. Note that the chosen codomain of a function is always a common set, e.g. $\mathbb{N}$, $\mathbb{R}$, ..., O.