A Quite Note On Set Theory

1 Sets

1.1 Basic Sets

Collected objects are often what mathematics focus on, so we have the notion of sets. A collection of objects "is called" a **set**. We care about the members of sets. Sets A and B "are called" **equal** if and only if they contain the same elements, by which I mean,

whenever $x \in A$, we have $x \in B$, and
whenever $y \in B$, we have $y \in A$.

Having sets, we hope to define operations on them.

DEFINITION. For sets A, B,
(i) $A \cap B := \{x \,|\, x \in A \text{ and } x \in B\}$;
(ii) $A \cup B := \{x \,|\, x \in A \text{ or } x \in B\}$;
(iii) $A \setminus B := \{x \,|\, x \in A \text{ but } x \notin B\}$.

Some properties are immediately.

PROPOSITION. Let S, T, U be sets.
(a) $(S \cap T) \cap U = S \cap (T \cap U)$.
(b) $(S \cup T) \cup U = S \cup (T \cup U)$.
(c) $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$.

PROOF. I only show (c). Let $x \in S \cup (T \cap U)$. Then $x \in S$ or $x \in T \cap U$. Our goal is to prove that $x \in S \cup T$ and $x \in S \cup U$. If $x \in S$, then $x \in S \cup T$ by definition of union. If $x \in T \cap U$, then $x \in T$ (definition of intersection). So $x \in S \cup T$ by definition of union.

Let $y \in (S \cup T) \cap (S \cup U)$. Then $y \in S \cup T$ and $y \in S \cup U$. We want to show that $y \in S$ or $y \in T \cap U$. If $y \in S$, then we finish the proof. Suppose that $y \notin S$. We want to show $y \in T$ and $y \in U$.

For $y \in T$, since $y \in S \cup T$ but $y \notin S$, by definition of union, we get that $y \in T$. Similarly, $y \in U$. Thus, by convention of set equality, we've shown the identity. □

Let A be a set. We call S a **subset** set of A if for any $x \in S$, it holds that $x \in A$. The **power set** of A is defined as

$$\wp(A) = \{S \,|\, S \subset A\}.$$

...

2 Relations

2.1 Basic Form

A **relation** between a set A and a set B is a subset R of $\wp(A \times B)$. By the notion of relation we hope to classify or build relationship between given sets. Note that we care about those relations with some specified properties.

Let R be a relation on A (i.e. a relation between A and itself).

(R) REFLEXIVITY: If $(x, x) \in R$ for any $x \in A$, then we say R is reflexive.

(S) SYMMETRY: Suppose that if $(x, y) \in R$ then $(y, x) \in R$. Then we say R is symmectric.

(A) ANTY-SYMMETRY: Suppose that if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. Then we say R is anty-symmectric.

(T) TRANSITIVITY: Suppose that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Then we say R is transitive.

EXAMPLE. $R_1 := \{(x, y) \in \mathbb{N} \times \mathbb{N} \,|\, x \leq y\}$, $R_2 := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \,|\, m \equiv n \,(\text{mod }) 5\}$, $R_3 := \{(p, q) \in \mathbb{N} \times \mathbb{N} \,|\, p \text{ divides } q\}$, and $R_4 = \{(S, T) \,|\, S \subset T,\ S, T \subset \mathbb{R}\}$ are examples of each type above.

For writing and reading convenience, we always write

$$xRy$$

instead of $(x, y) \in R$.

2.2 Equivalence Relations

...

3 Cardinality

...

4 Cardinal Numbers

...

5 Ordered Sets

...

6 Axiomization

A precise development of Set Theory requires some axioms, which can be viewed as a start of the theory. Every proposition is duducted from either axioms or from lower-leveled propositions.

...