Mathematical Entertainments

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This column is interested in publishing mathematical material which satisfies the following criteria, among others:

1. It should not require technical expertise in any specialized area of mathematics.

2. The topics treated should when possible be comprehensible not only to professional mathematicians but also to reasonably knowledgeable and interested nonmathematicians.

We welcome, encourage and frequently publish contributions from readers. Contributors who wish an acknowledgement of submission should enclose a self-addressed postcard.

Has this ever happened to you? You've just finished patiently trying to explain some beautiful result of pure mathematics to a group of nonmathematicians, hoping that you've conveyed something of the flavor of this pearl of truth and beauty, and then after a pause someone says, "Yes, but what has any of this got to do with everyday life?". After much thought I've decided the correct response is to say "Nothing. That's what's so nice about it. After all, Every Day Life is often a drag, so we do mathematics for the same reason we listen to music or ski down a mountain, to get away from and above and beyond Every Day Life."

But now I must admit that every once in a while it works out the other way and EDL turns out to be a source of unexpectedly interesting mathematics. A nice example of this is the following item, written by guest columnist John H. Halton.

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The Shoelace Problem

John H. Halton

In a number of discussions of how shoes should be laced, it became apparent that no one seemed to have the definitive answer. Shoes were laced and relaced, passions flared, and shoes were even thrown. . . . The author decided that an appeal to mathematics was indicated.

This problem is a restriction of the Traveling Salesman Problem. We are given a set of $2(n + 1)$ points (the *laceholes* or *eyelets*) arranged in a bi-partite lattice, as shown in Figure 1.

The problem is to find the shortest path from A_0 to B_0, passing through every eyelet just once, in such a way that points of the subsets

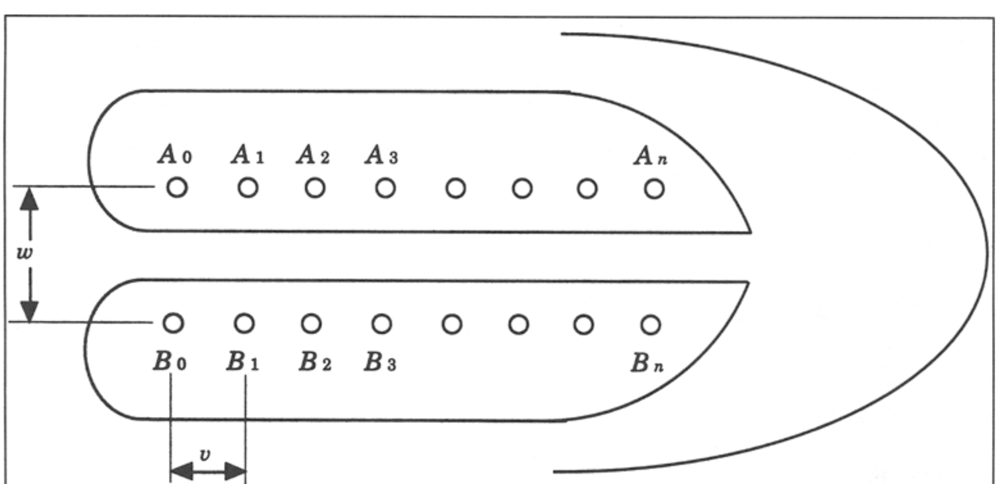

Figure 1. The shoe (a schematic).

$$A = \{A_0, A_1, A_2, \ldots, A_n\} \quad \text{and}$$
$$B = \{B_0, B_1, B_2, \ldots, B_n\} \qquad (1)$$

alternate in the path.

Three standard lacing strategies are shown in Figures 2–4.

For the American (AM) style, as in Figure 2, if n is *odd*, the lacing is

$$\begin{aligned} &A_0 \to B_1 \to A_2 \to B_3 \to A_4 \to \cdots \\ &\to A_{n-1} \to B_n \to A_n \to B_{n-1} \to A_{n-2} \to B_{n-1} \to \cdots \\ &\to A_3 \to B_2 \to A_1 \to B_0; \end{aligned} \qquad (2)$$

if n is *even*, the lacing is, similarly,

$$\begin{aligned} &A_0 \to B_1 \to A_2 \to B_3 \to A_4 \to \cdots \\ &\to A_{n-2} \to B_{n-1} \to A_n \to B_n \to A_{n-1} \to B_{n-2} \to \cdots \\ &\to A_3 \to B_2 \to A_1 \to B_0; \end{aligned} \qquad (3)$$

and it is easily verified that, in either case, the total length of lace used is

$$L_{AM} = L_{AM}(n, v, w) = w + 2n\sqrt{v^2 + w^2}. \qquad (4)$$

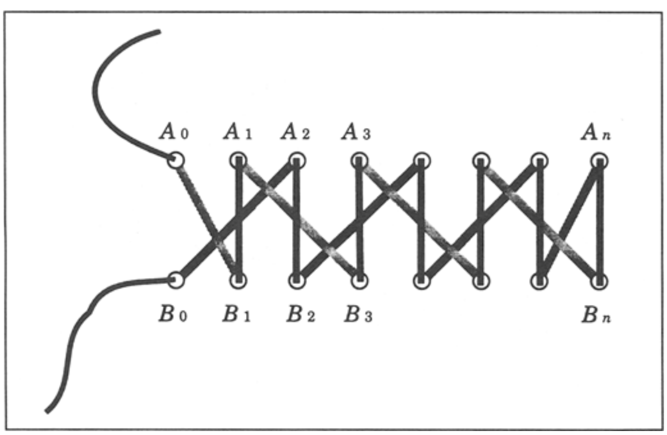

Figure 2. American zig zag lacing.

For the European (EU) style, as in Figure 3, when n is *odd*, the lacing is

$$\begin{aligned} &A_0 \to B_1 \to A_1 \to B_3 \to A_3 \to \cdots \\ &A_{n-2} \to B_n \to A_n \to B_{n-1} \to A_{n-1} \to B_{n-3} \to \cdots \\ &\to B_2 \to A_2 \to B_0; \end{aligned} \qquad (5)$$

when n is *even*, the lacing is, similarly,

$$\begin{aligned} &A_0 \to B_1 \to A_1 \to B_3 \to A_3 \to \cdots \\ &\to A_{n-1} \to B_n \to A_n \to B_{n-2} \to A_{n-2} \to B_{n-4} \to \cdots \\ &\to B_2 \to A_2 \to B_0; \end{aligned} \qquad (6)$$

and, with a little more thought, we see that, in both cases, the total length of lace is

$$\begin{aligned} L_{EU} &= L_{EU}(n, v, w) \\ &= nw + 2\sqrt{v^2 + w^2} + (n-1)\sqrt{4v^2 + w^2}. \end{aligned} \qquad (7)$$

For the shoe shop (SS) style, as in Figure 4, the lacing is

$$\begin{aligned} &A_0 \to B_n \to A_n \to B_{n-1} \to A_{n-1} \to \cdots \\ &\to B_3 \to A_3 \to B_2 \to A_2 \to B_1 \to A_1 \to B_0 \end{aligned} \qquad (8)$$

and we find that the total length is

$$\begin{aligned} L_{SS} &= L_{SS}(n, v, w) \\ &= nw + n\sqrt{v^2 + w^2} + \sqrt{n^2v^2 + w^2}. \end{aligned} \qquad (9)$$

We can generalize the situation as follows. Let α and β denote permutations of $\{1, 2, 3, \ldots, n\}$:

$$\begin{aligned} \alpha &= \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \\ \beta &= \{\beta_1, \beta_2, \ldots, \beta_n\}. \end{aligned} \qquad (10)$$

To them will correspond the lacing

$$\begin{aligned} &A_0 \to B_{\beta_1} \to A_{\alpha_1} \to B_{\beta_2} \to A_{\alpha_2} \to B_{\beta_3} \to \cdots \\ &\to A_{\alpha_{n-1}} \to B_{\beta_n} \to A_{\alpha_n} \to B_0, \end{aligned} \qquad (11)$$

and this will have total length

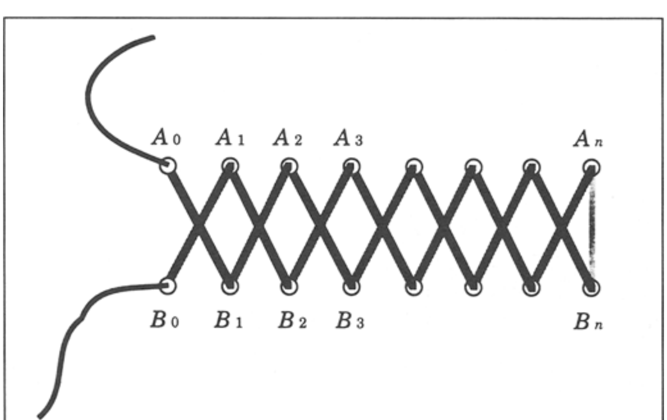

Figure 3. European straight lacing.

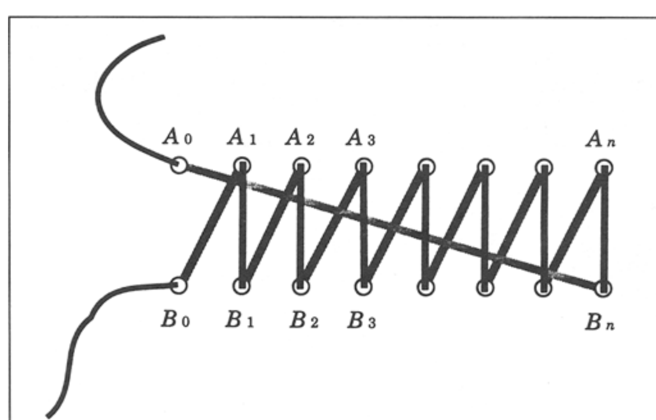

Figure 4. Shoe-shop quick lacing.

$$L = \sqrt{\beta_1^2 v^2 + w^2} + \sqrt{(\alpha_1 - \beta_1)^2 v^2 + w^2} + \sqrt{(\beta_2 - \alpha_1)^2 v^2 + w^2} + \sqrt{(\alpha_2 - \beta_2)^2 v^2 + w^2} + \cdots + \sqrt{(\beta_n - \alpha_{n-1})^2 v^2 + w^2} + \sqrt{\alpha_n^2 v^2 + w^2}. \quad (12)$$

For the three special lacings shown above, the particular permutations are

$$\alpha_{AM} = \{\text{all even numbers increasing; then all odd numbers decreasing}\},$$
$$\beta_{AM} = \{\text{all odd numbers increasing; then all even numbers decreasing}\}; \quad (13)$$

$$\alpha_{EU} = \beta_{EU} = \beta_{AM}, \quad (14)$$

$$\alpha_{SS} = \beta_{SS} = \{\text{all numbers decreasing}\}. \quad (15)$$

The simplicity of these permutations is indeed remarkable.

THEOREM 1. *If $v = 0$ or $w = 0$, for all positive n,*

$$L_{AM} = L_{EU} = L_{SS}. \quad (16)$$

If $v \geq 0$ and $w \geq 0$,

$$L_{AM}(1, v, w) = L_{EU}(1, v, w) = L_{SS}(1, v, w), \quad (17)$$

and, if $v > 0$ and $w > 0$,

$$L_{AM}(2, v, w) < L_{EU}(2, v, w) = L_{SS}(2, v, w). \quad (18)$$

Finally, if $v > 0$, $w > 0$, and $n > 2$,

$$L_{AM} < L_{EU} < L_{SS}. \quad (19)$$

This theorem can be proved, using (4), (7), and (9), by the careful analysis of cases and elimination of radicals. The proof is left as an exercise for the reader. (It is given by the author in a technical report [1].)

The Lattice Representation

Let us make a lattice of alternating parallel, equidistant sets A and B, as shown in Figure 5. Given any lacing $\mathcal{L}$, we can represent it, as is shown for our three standard examples, by a polygonal (piecewise straight) line L moving always downward across the new lattice, visiting the eyelet points only once each.

The first line segment in the order of lacing, $A_0 \to B_{\beta_1}$, is unchanged; the next, $B_{\beta_1} \to A_{\alpha_1}$, is replaced by its mirror image in the original B line; the next $A_{\alpha_1} \to B_{\beta_2}$, is moved downward by two lattice intervals, parallel to itself (i.e., it is a twice-repeated mirror image), and so on; the last segment, $A_{\alpha_n} \to B_0$, returns to the image of B_0 in the B line displaced downward by $2n$ intervals. Clearly, the total length of the representation L will equal the original total length of L the lacing $\mathcal{L}$ itself.

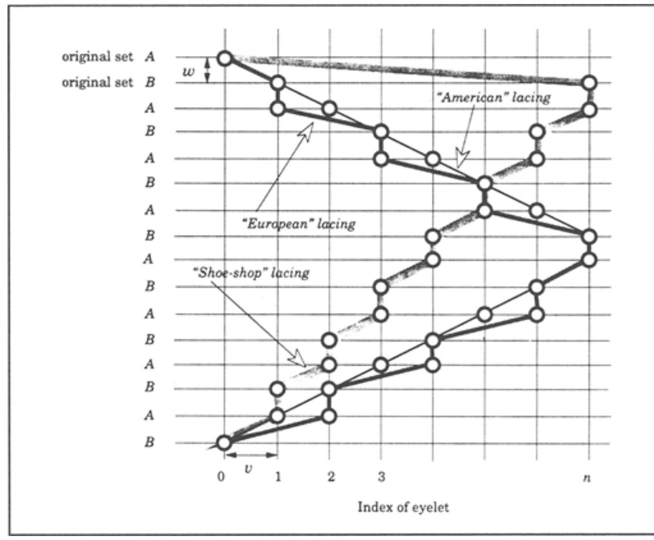

Figure 5. Lattice representation of the three standard lacings.

That the "American" (AM) lacing is better than the "European" (EU) lacing is now immediately apparent, by a straightforward application of the triangle inequality (see Figure 6).

The two representations, L_{AM} and L_{EU} coincide in several places. Where they differ, replicas of a triangle PQR occur, and it is clear that $PR < PQ + QR$, so that the first inequality in (19) follows, without further algebra!

That the EU lacing is better than the SS lacing is a little harder to prove (see Figure 7). First, we observe that the representations L_{EU} and L_{SS} have in common just *two* diagonal segments, moving by one lattice interval in both directions (slopes $\pm$ w/v), and n (vertical) segments, moving by one vertical lattice interval w only. If we omit all of these common intervals, shifting the sep-

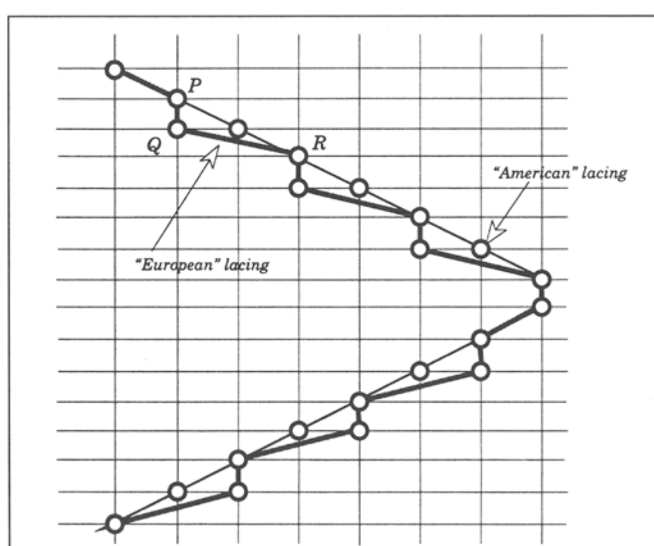

Figure 6. Comparison of AM and EU lacing.

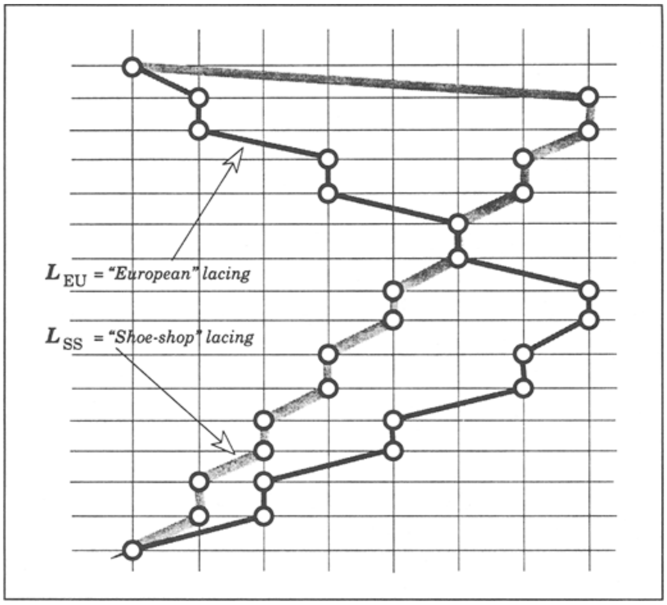

Figure 7. Comparison of EU and SS lacing.

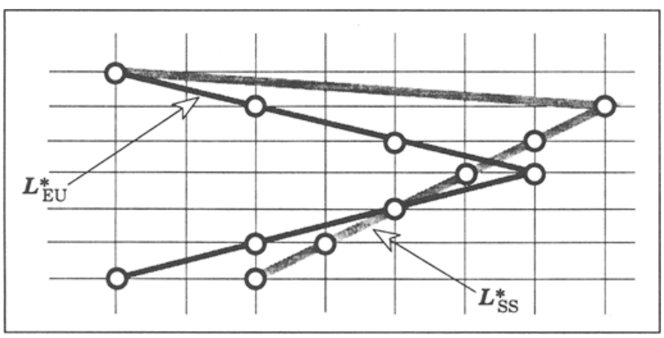

Figure 8. Comparison of EU and SS lacing—reduced representations.

arated lower segment upward (and in the first two cases, sideways also), parallel to themselves, to rejoin the upper segment, and thus *subtracting equal lengths from each representation*, we obtain reduced representations, L^*_{EU} and L^*_{SS}. The result is shown in Figure 8. Each representation now consists of a singly-broken line (just two successive line segments—a *zig* and a *zag*).

Now perform the "reflection trick" again, this time in the horizontal coordinate direction, so that the leftward segment of each representation is reflected about the vertical. The resulting representation lines are denoted by L^{**}_{EU} and L^{**}_{SS} (see Figure 9).

We can now simply observe that L^{**}_{EU} is just a single straight segment UV, whereas L^{**}_{SS} consists of two straight segments, UW and WV, so that, again by the triangle inequality, (19) clearly holds.

Optimization

We adopt the lattice representation described above (see Figures 5–7) and apply the "reflection trick" to the part of the path from B_n to B_0. The form of the path corresponding to a typical general lacing is illustrated in Figure 10. The path L_{AM} corresponding to the AM lacing is also shown. In this particular example, as before, $n = 7$ and the lacing is

$$A_0 \to B_2 \to A_7 \to B_4 \to A_6 \to B_1 \to A_1 \to B_3 \to A_3 \to B_6 \to A_5 \to B_5 \to A_4 \to B_7 \to A_2 \to B_0. \quad (20)$$

Its length is [compare (12) and collect similar radicals]

$$L = 3w + 2\sqrt{v^2 + w^2} + 4\sqrt{4v^2 + w^2} + 3\sqrt{9v^2 + w^2} + 3\sqrt{25v^2 + w^2}. \quad (21)$$

In general, let the lacing have total length

$$L = \sum_{k=-n}^{n} N_k \sqrt{k^2v^2 + w^2}, \quad (22)$$

where, clearly,

$$\sum_{k=-n}^{n} N_k = 2n + 1 \quad (23)$$

is the net total number of downward displacements (i.e., the number of steps, since each step has a downward displacement by one lattice interval w), and

$$\sum_{k=-n}^{n} kN_k = 2n \quad (24)$$

is the net total number of rightward displacements by one lattice interval v. For the AM lacing, it is clear that

$$N_0 = 1, N_1 = 2n, \quad \text{all other } N_k = 0. \quad (25)$$

THEOREM 2. *The AM lacing has the shortest possible total length L, and it is the unique optimum lacing.*

Proof. Let **L** be the reflected representation of an arbitrary lacing $\mathscr{L}$, and let L be its total length.

(i) If $N_0 \geq 1$, let us remove any one corresponding (vertical) step from $\mathscr{L}$, and let us remove the sole vertical step from $\mathscr{L}_{AM}$, rejoining the separated pieces of the representations by parallel displacement, as before; then

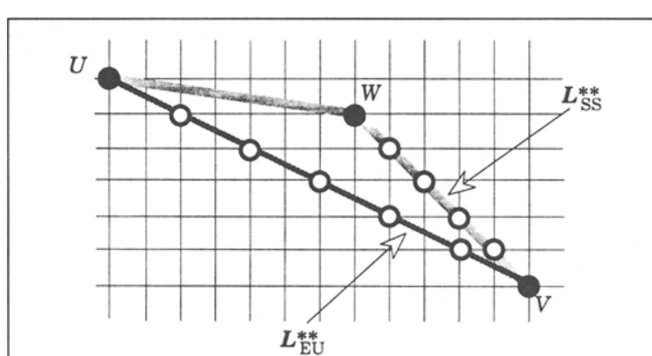

Figure 9. Comparison of EU and SS lacing—reflected representations.

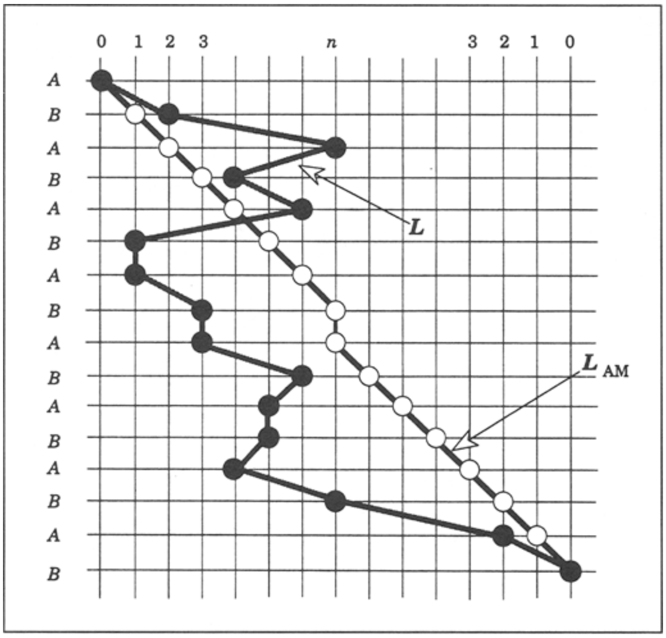

Figure 10. General lacing—reflected representations

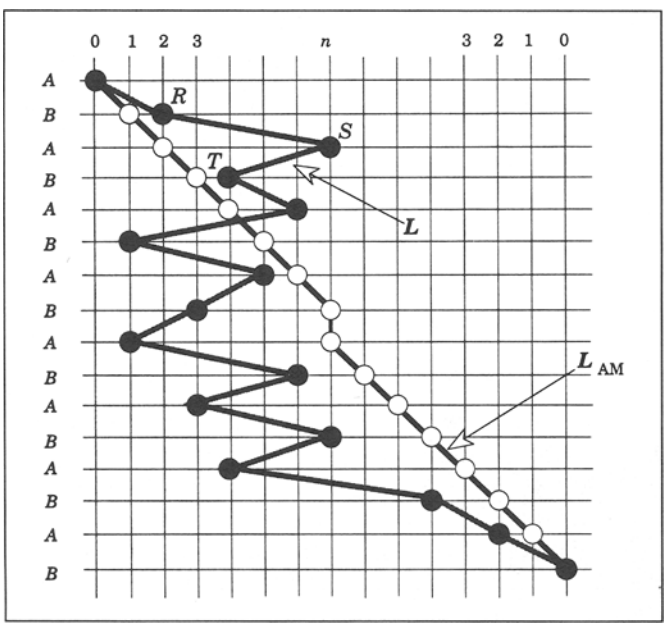

Figure 11. Case of $N_0 = 0$—no vertical segment.

the two new representations, $L^{\dagger}$ and $L^{\dagger}_{\mathrm{AM}}$, still share their end points, and both lengths are just w less than they were. Now $L^{\dagger}_{\mathrm{AM}}$ is clearly minimal, being the straight line connecting these end points. Therefore, for all $\mathcal{L}$,

$$L_{\mathrm{AM}} \leq L. \tag{26}$$

(ii) Suppose now that $N_0 = 0$. This is illustrated in Figure 11.

It cannot be that $N_k > 0$ only for positive values of k; for then, by (23) and (24), we would have that

$$\sum_{k=1}^{n} kN_k - \sum_{k=1}^{n} N_k = N_2 + 2N_3 + \cdots + (n-1)N_n = 1, \tag{27}$$

which is impossible, since all $N_k \geq 0$. Therefore, there is at least one step with a negative (*leftward*) horizontal displacement, and thus there is a first leftward step, ST, in the downward order. It obviously *cannot* be either the *first* or the *last* step of the representation. Hence, it is preceded by a *rightward* step, RS, forming an angle pointing to the right.

Now (see the enlarged detail of Figure 12), let F and G be the respective lattice points in which the vertical

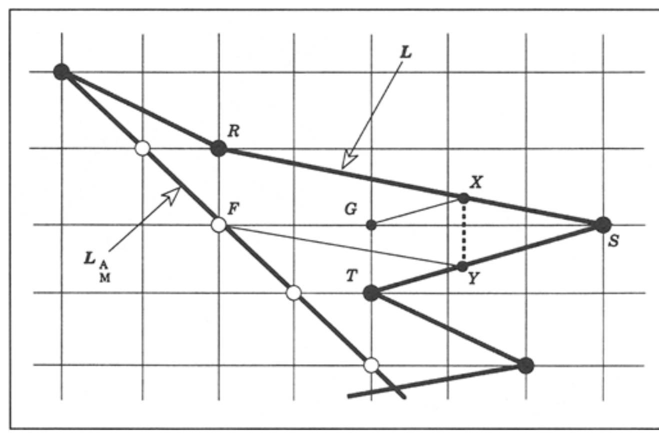

Figure 12. Magnified detail of Figure 11.

lines through R and T meet the horizontal line through S. Then

$$|FR| = |GT| = w \tag{28}$$

and

$$|FS| \geq v \quad \text{and} \quad |GS| \geq v. \tag{29}$$

Through G, draw a line parallel to TS, and let it meet RS (as it must) at X. Now draw a vertical line (parallel to RF) through X to meet TS at Y. Clearly, $XYTG$ is a parallelogram, and therefore $|XY| = |GT| = w$, by (28).[1]

Thus we can replace the polygonal segment RST of the representation L by the polygonal segment $RXYT$, and by the triangle inequality,

$$|XY| < |XS| + |SY|; \tag{30}$$

[1]Note, too, that $XYFR$ is also a parallelogram, since the opposite sides, XY and RF, are equal and parallel.

so that the modified representation $L^{\perp}$ say, is *shorter* than L. But now $L^{\perp}$ *has* a vertical segment of length w; so, by the same argument as in case (i), the inequality (26) prevails.

NOTE: The representative polygonal line $L^{\perp}$ is, generally, *not a representation of any lacing*, since it does not, in general, join lattice points; but this does not matter, since, at this stage of the argument, we are only concerned with the *length* of the line.

We have now proved that, if $\mathcal{L}_{MIN}$ is any lacing of minimal length, then it and its (horizontally reflected) representation L_{MIN} will have a total length equal to that of the AM lacing, that is, by (4),

$$L_{MIN} = L_{AM} = w + 2n\sqrt{v^2 + w^2}. \qquad (31)$$

(iii) Finally, we prove the *uniqueness* of the optimal lacing $\mathcal{L}_{MIN}$. The arguments presented in cases (i) and (ii) show that any minimal lacing $\mathcal{L}_{MIN}$ will satisfy (25); that is, its (horizontally reflected) representation L_{MIN} will have $2n$ straight segments, moving diagonally down-and-to-the-right by one lattice interval, and one vertical segment. However, the *position* of this vertical segment in the chain does not matter to the total length L_{MIN}, as is indicated in (31).

Nevertheless, since L_{MIN} is not just any lattice polygon, but the representation of a *lacing*, it *must* pass through the vertical lattice line corresponding to index n just twice (corresponding to the eyelets A_n and B_n), and this is the *only* lattice line which is *not* duplicated by the reflection transformation, since it *is* the reflection line. Therefore, since the representation moves monotonely right (i.e., never to the left), the solitary vertical segment is constrained to be precisely in the index n position, as in L_{AM}. This completes the proof of Theorem 2. □

Reference

1. The Shoelace Problem, Department of Computer Science Technical Report No. 92-032 (1992), University of North Carolina at Chapel Hill.

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