Mathematical Entertainments

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For the general philosophy of this section see Vol. 13, no. 1 (1991). Contributors to this column who wish an acknowledgment of their contribution should enclose a self-addressed postcard.

We All Make Mistakes

Some of us more than others, so it may be comforting to realize that occasionally even the great mathematicians have published erroneous results. Readers are invited to contribute examples of this phenomenon, where a mistake should mean not just a gap in a proof or a case of meaning one thing and writing another, but rather an explicit assertion that is false. Are there any such examples in the work of Gauss, I wonder? R. M. Robinson has called my attention to a lapse by Minkowski in which he asserts that the difference set of a tetrahedron is an octahedron. In his 1906 paper "Dichteste gitterförmige Lagerung kongruenter Körper," *Nachrichten der König. Gesellschaft der Wissenschaften zu Göttingen* 5(1904), 311–355, Minkowski writes

> For example, if K is a tetrahedron, then $\frac{1}{2}(K + K')$ becomes an octahedron with faces parallel to the faces of the tetrahedron.

Here K' is "the reflection of K with respect to the point O."

It's a curious sort of error since the polyhedron in question clearly has 12 rather than 6 vertices, namely all pairs $a_i - a_j$, $i \neq j$, where the a_i are the vertices of the tetrahedron. The correct polyhedron is in fact the convex hull of the midpoints of the 12 edges of a cube, so it has eight triangular and six square faces.

Paradoxes and a Pair of Boxes

What is a paradox? Perhaps the best-known examples in mathematics are Russell's paradox and the Banach-Tarski paradox, but it should be noted that the nature of these two results is very different. The Banach-Tarski Theorem is considered paradoxical because it shows that sets can behave in a way very different from our intuitive notions about them. Russell's paradox, on the other hand, shows that starting from what seem to be plausible axioms one can arrive at a contradiction. The proper term for this is not paradox but *antinomy*, which, according to Webster, is "a contradiction between two apparently equally valid principles or between inferences correctly drawn from such principles," while a paradox is "a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true."

In the examples to follow the first two are antinomies and the last two are paradoxes.

1. *The other box.* You are presented with two boxes, each containing a certain amount of money that has been placed there by the following rule. A fair coin was tossed until it fell tails. If n heads were tossed then one of the boxes contain 3^n dollars and the other 3^{n+1}. You are allowed to open one of the boxes and count its contents. You may then either pocket this money or switch and take the money in the as yet unopened box. What should you do? Well, clearly if the box you open contains one dollar you should take the three dollars in the other box. Now suppose the box you open contains 3^n dollars. Then one easily sees that the other box will contain 3^{n-1} or 3^{n+1} dollars with respective probabilities ⅔ and ⅓, so your expectation from switching is

$$\frac{2}{3}3^{n-1} + \frac{1}{3}3^{n+1} = \frac{11}{9}3^n > 3^n,$$

so you maximize your expected winnings by switching; so, assuming you are an expectation maximizer, this is what you will do. (The debate as to whether expectation maximizing is "reasonable" is of course beside the point, since we are concerned here only with the mathematics and not with its implications for behavior.) But now, since you know in advance you will always switch, there is no point in wasting time opening and counting: you should simply choose "the other box" to begin with, and the same argument then shows that whichever box you choose, you would have been better off expectation-wise to have chosen the other.

Readers will have noticed that this game has a definite Petersburgian flavor in that the expected win-

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nings in the game are infinite. The novelty of this variant is the fact that it seems to lead to a contradiction and thus we are dealing with an antinomy rather than a mere paradox.

2. *Beat the house.* A somewhat similar example was told to me by Lester Dubins, who is uncertain as to its origin. In a certain casino one can play the following game. The house posts a positive integer n. In this game it is you the customer who is invited to toss the fair coin until it falls tails. If you tossed $n - 1$ times then you pay the house 8^{n-1} dollars, but if you tossed $n + 1$ times you win 8^n dollars from the house. In all other cases the payoff is zero. Since the probability of tossing exactly n times is $1/2^n$, your expected winnings are $8^n/2^{n+1} - 8^{n-1}/2^{n-1} = 4^{n-1}$ for $n > 1$, and 2 for $n = 1$, so your expected gain, which is the house's expected loss, is positive. But now it turns out that the house arrived at the number n by tossing that same fair coin and counting the number of tosses up to and including the first tails. Thus, you and the house are behaving in a completely symmetric manner. Each of you tosses the coin and if the number of tosses happens to be the consecutive integers n and $n + 1$, then the n-tosser pays the $(n + 1)$-tosser 8^n dollars. But we have just seen that the game is to your advantage as measured by expectation no matter what number the house announces. How can there be this asymmetry in a completely symmetric game?

3. *The other box again.* (A mild modification of an example due to David Blackwell.) This time the boxes contain not money but each box contains an integer (perhaps printed on a card), and the only thing you know about them is that they are distinct. You draw one of them at random and are then supposed to guess whether the other is higher or lower. Is there anything you can do so that you will have a better than even chance of guessing correctly? Surprisingly the answer is yes, provided you have some mechanism for randomizing as, for example, a true coin to toss. To be specific, suppose you have a spinner like those used in children's board games. You should proceed to spin and then record the angle θ between the initial and final position of the pointer. Now draw your number and guess higher or lower according as $\cot \theta/2$ is greater or less than the number you drew. Let us assume for convenience that θ has the uniform distribution on $[0,2\pi)$. Claim: If the two numbers are $p > q$, then the probability that you will guess correctly is $½ + (\cot^{-1}q - \cot^{-1}p)/2\pi$. Namely, $q < \cot \theta/2 < p$ if and only if $2 \cot^{-1}p < \theta < 2 \cot^{-1}q$ and from uniformity this has probability $\gamma = (\cot^{-1}q - \cot^{-1}p)/\pi$. In this case because of the guessing rule you will be right no matter which number you draw. In the other cases where θ is either greater or less than both p and q your probability of guessing right is one-half, since you are equally likely to draw p or q. Thus your probability of a correct guess is $½(1 - \gamma) + \gamma = 1/2 + \gamma/2$ as claimed.

Mathematically the argument above is air-tight but it raises some interesting philosophical questions about the applicability of probability theory in decision making. For example, suppose you don't have a spinner but are wearing a watch, the old-fashioned nondigital kind, and choose θ to be the angle between the minute and hour hand in order to make your guess. Is this in any sense a kind of randomization and if not why not? Or, suppose instead of numbers the boxes contain stones of different weights and you have a balance so you can compare weights but no scale for making measurements. You draw a stone and must guess whether the other is heavier or lighter. Is there anything you can do? This might lead one to think that randomization is possible only for *quantitative* comparisons, meaning that one must associate numbers with the objects drawn, but this need not be true either. Suppose, for example, the boxes contain slices of pie. Then a spinner is just what you need. Guess bigger or smaller depending on whether the spinner angle is greater or less than the angle of the slice you draw. This can be determined by direct visual comparison. There is no need for a protractor and numbers are not involved in any way.

4. *The other's number.* This time the integers in the boxes are positive and consecutive. Each player draws one and is supposed to find out the opponent's number by the following procedure. The players are equipped with a blank card and a pencil. If at any time a player knows her opponent's number, she writes it on her blank card and wins the game. If neither player knows the other's number, they exchange blank cards and start over. The assertion is that, with perceptive players, this game will terminate. More precisely we have

THEOREM. *If the two numbers are n and $n + 1$, then the player holding n will win after $n - 1$ exchanges.*

The proof is by induction. If $n = 1$ then the player holding 1 will know the opponent's number is 2 and the game ends with no exchanges. Now assume the conclusion is true up to n and suppose the lower number is $n + 1$. Then the player holding this number knows that if her opponent holds n he will end the game after the $(n - 1)$st exchange (induction hypothesis), so when he doesn't do this she knows after the exchange that he must hold $n + 2$ and she wins.

The paradoxical point is this: suppose the numbers held are, say, 72 and 73. Then neither player knows the other's number and both are aware of their opponent's ignorance so they know for sure that the first stage of the game will be an exchange of cards; so when this indeed takes place they have apparently gained no new knowledge, yet since the game is now one step closer to termination, something must have changed. What was it?

Problems

Supporting cords of convex sets (91-2) by Serge L. Tabacnikov (Moscow, USSR).

Let A,B be plane convex sets with $A \subset \text{int } B$. Prove there are at least two cords of B that are tangent to A at their midpoints.

Boomerang problem: Quickie (91-3) (origin unknown to column editor; information would be appreciated).

A *boomerang* is a nonconvex quadrilateral. Prove that it is impossible to tile a convex polygon with a finite number of (not necessarily congruent) boomerangs.

Solutions

The modified Fermat problem: Quickie (90-7) by Flejberk Jaroslav (Pardubice, Czechoslovakia).

For any two relatively prime positive integers n and k, show that the equation $x^n + y^n = z^k$ has a solution in positive integers x, y and z.

Solution by B. M. M. de Weger (University of Twente, The Netherlands).

Since n and k are coprime, there are positive integers a,b with $bk - an = 1$. Then $x = y = 2^a$, $z = 2^b$ is a solution.

Two contributions from Lee Sallows

First the following:

"This computer-generated sentence contains two hundred forty-seven letters: four *a*'s, one *b*, four *c*'s, five *d*'s, forty-four *e*'s, nine *f*'s, three *g*'s, seven *h*'s, eleven *i*'s, one *j*, one *k*, three *l*'s, two *m*'s, twenty-nine *n*'s, nineteen *o*'s, two *p*'s, one *q*, fourteen *r*'s, thirty-one *s*'s, twenty-five *t*'s, seven *u*'s, eight *v*'s, seven *w*'s, two *x*'s, six *y*'s, and one *z*."

Now that you've had a chance to verify the correctness of the sentence you may wonder how the computer generated it. Here is Sallows' description of how it's done.

"The algorithm that generated the above sentence implements an iterated function. Starting with a similar text, but using randomly selected totals, its true letter frequencies are determined and then substituted for these, the new version then furnishing the argument for the next iteration, and so on. The result is a series of approximations tending toward the goal. I like to picture this process as a machine that takes sentences as input and yields sentences as output, the latter coupled back to the input via a feedback loop. This makes it easier to see that a self-descriptive sentence is effectively a virus able to subvert the machine so as to get itself perpetually reproduced. Such a sentence has only to appear once at the input in order to trigger a closed loop of period 1 and thus be regurgitated ad nauseam (if you see what I mean). The only trouble is, there are still other viruses that will probably infect the machine first! These are the sentence chains of longer period, in any one of which it may easily become ensnared, and thus be prevented from converging onto a self-descriptor. How can we immunize the machine against such interloopers?

My answer is a modified machine that will scramble possible cycles through performing non-repetitively: Instead of correcting every total on every pass, I have it correct a single total chosen at random each time. Now no recurrent cycle can survive such irregular exchanges, except of course in the special case where the totals remain unchanged because already correct: a self-descriptor! [Note here the complete analogy with a neural network settling into a stable solution state, while avoiding latch-up in pseudo-solution states through "jiggling."] In fact the "random" selection need not be truly random, provided only that the repetition period of its own pattern be longer than that of any possible loop the machine may fall into. Hence, any conventional pseudo-random number generator serves well. A few million iterations (mutations) normally suffice to evolve (naturally select) a viable solution (virus) provided one exists. If not we can try again with a modified text."

Sallows' second contribution involves neither letters nor numbers and is presented below without comment.

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