Mathematical Entertainments

David Gale*

For the general philosophy of this section see Vol. 13, No. 1 (1991). Contributors to this column who wish an acknowledgement of their contributions should enclose a self-addressed postcard.

We devote the column this time to some recent results on a pair of fairly well-known problems of recreational mathematics that have been around for quite a while. The first is the problem of tiling surfaces by unequal squares, the second that of devising fair procedures for dividing a cake. The results on tiling are rather definitive, whereas the work on cake-cutting is still in a rather formative stage.

Tiling of Surfaces by Unequal Squares

The question is, or rather was, which rectangles can be tiled by squares no two of which have the same size. Figure 1 is an example of a 32 × 33 rectangle which is tiled by 9 such squares. This example, apparently discovered by Moron in 1925, appears in Ball's *Mathematical Recreations* and Steinhaus's *Mathematical Snapshots*. In 1940, Tutte, Brooks, Smith, and Stone (*Duke Math J.* 7 (1940), 312–340) were able to show that this is the "smallest" such example, meaning that no rectangle can be tiled in this way by fewer than 9 squares. They also showed, however, that there is exactly one other rectangle, 61 × 69, shown in Figure 2, which can also be tiled by 9 squares. The authors were actually seeking and eventually found a square which could be tiled in this way. For an entertaining exposition, see the chapter by Tutte, "Squaring the Square," in Martin

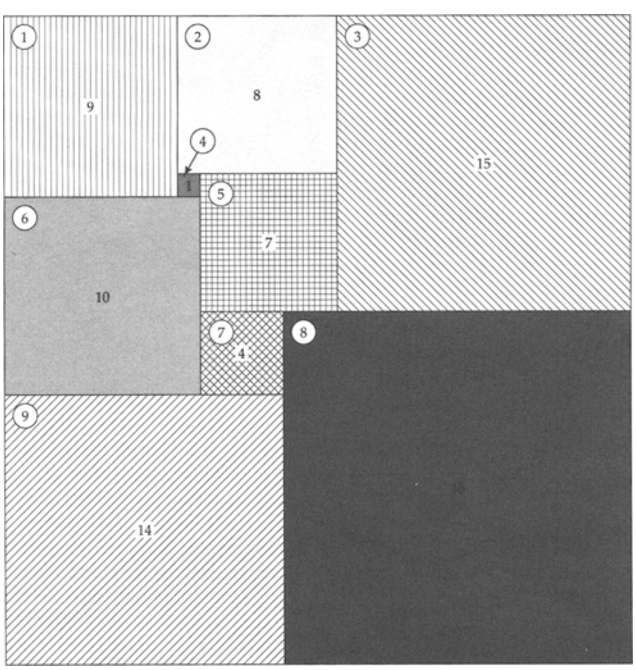


Figure 1

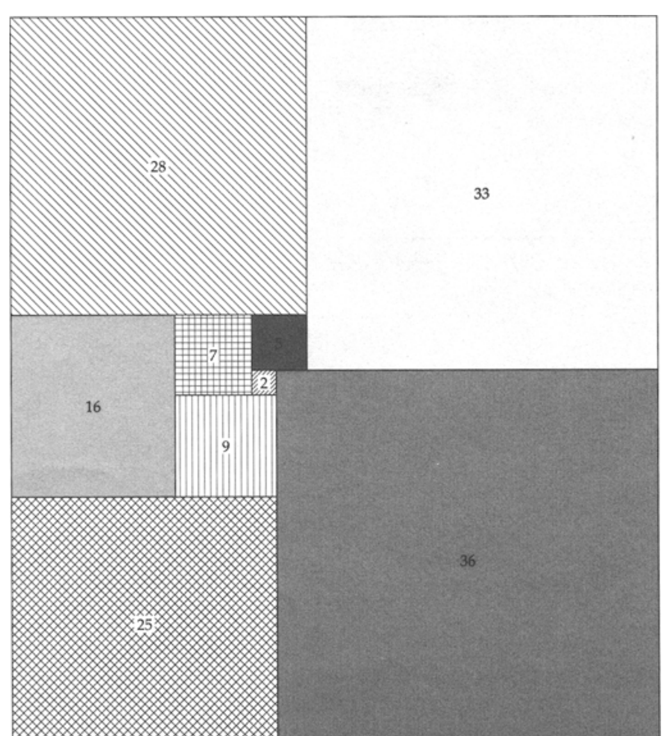


Figure 2

* **Column editor's address:** Department of Mathematics, University of California, Berkeley, CA 94720 USA.

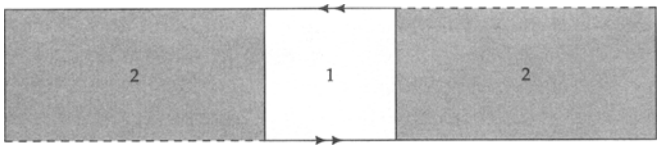


Figure 3

Gardner's second *Scientific American Book of Puzzles and Diversions*, Simon and Schuster, 1961.

Now, such a "squaring" of a rectangle can be converted in a trivial way to a squaring of the cylinder, torus, Möbius strip, or Klein bottle by the usual identification of opposite sides, but there are also nontrivial squarings of these other surfaces, even *simple squarings*, meaning squarings in which there is no subset of tiles whose union is a rectangle. Until recently, however, it was not known whether there might be squarings of these surfaces requiring fewer than 9 squares. Then, in 1991, Bracewell found a squaring of the Möbius strip using only 8 squares. Very recently the question has been completely settled by S. J. Chapman. Perhaps the simplest but most surprising result is that a 1×5 Möbius strip can be tiled by 2 squares, as becomes obvious from Figure 3. To accommodate this example, one must extend the notion of a tiling to allow the mapping of the squares into the surface to self-intersect on their boundaries.

Chapman shows that there are no 3- or 4-squarings of the strip but there is a unique 5-squaring (Fig. 4).

For cylinders, the situation is interesting. Again it turns out that 9 squares are necessary. There are exactly two nontrivial 9-squarings of the cylinder, and these use exactly the squares of Figures 1 and 2 but in a different arrangement. The tiling corresponding to Figure 1 is shown in Figure 5. Note, for example, that the squares of size 10 and 15 are disjoint in the rectangle but they are contiguous on the cylinder.

The cases considered so far involve surfaces with boundary, which forces one to orient the squares with one side parallel to the boundary. This is no longer the case with the torus and Klein bottle. If one allows arbitrary orientation, then, in fact, any two squares can tile some torus. Namely, let the squares have sides a and b and consider the torus obtained from identifying opposite sides of a square of side c, where $c^2 = a^2 + b^2$. Figure 6 shows how to do it and at the same time provides a new (?) proof of the Pythagorean Theorem.

A more symmetrical representation is given in Figure 7.

If one allows only tiles which are parallel to the sides of the square, then it turns out there are no nontrivial 9-squarings of the torus. Any squaring of the Möbius strip gives a squaring of the Klein bottle. For 6 or fewer tiles these are the only ones, but in the case of 7 or 8 tiles this is not known. Also it is not known whether there are tilings of the Klein bottle in which the tiles need not be parallel to the sides of the big square.

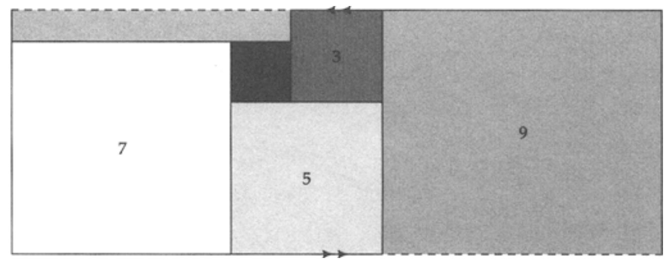


Figure 4

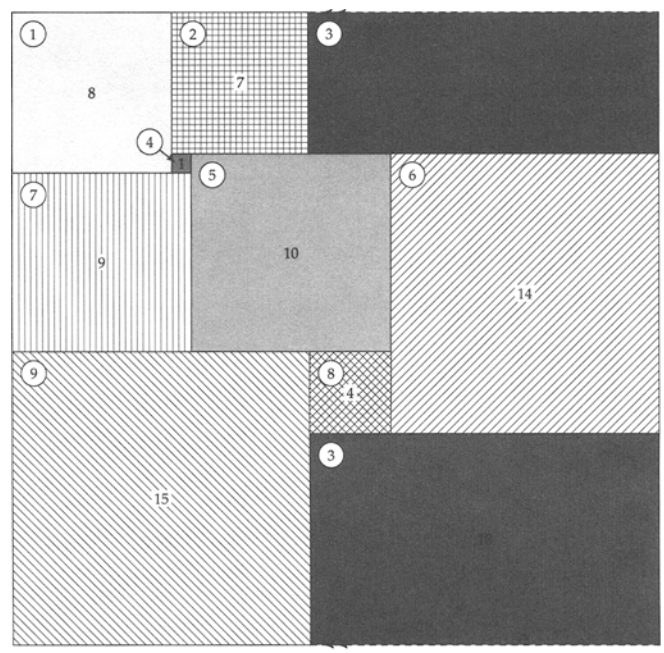


Figure 5

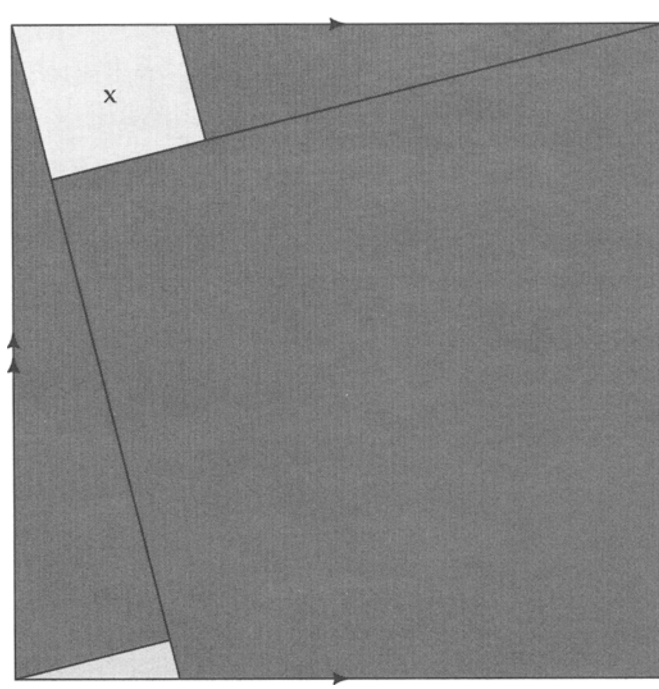


Figure 6

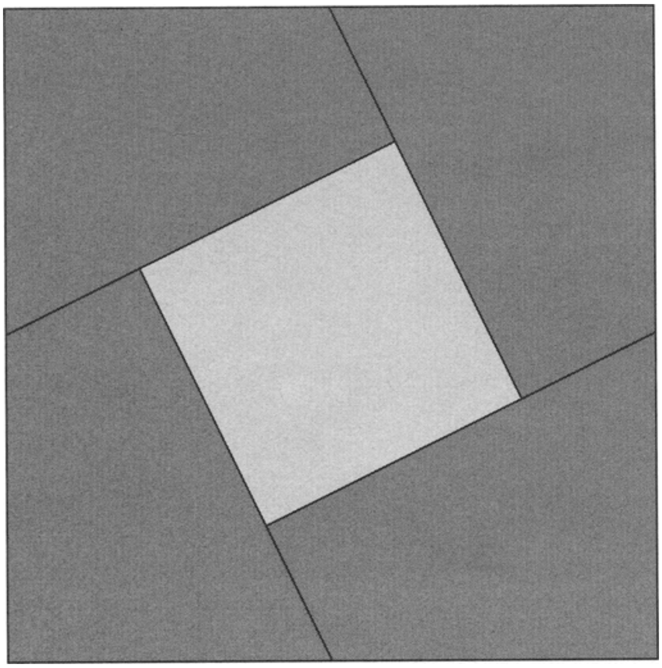


Figure 7

Chapman's techniques for these problems are quite different from and simpler than those of Tutte, *et al.* and depend on a clever encoding of tilings by matrices of 0's, 1's, and -1's.

Dividing a Cake

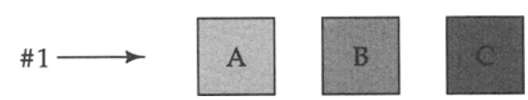
A cake is to be divided between *n* of us. We have different tastes. Some of us like the frosting, others are partial to the chocolate filling, etc. Is there a way of giving each of us a piece of the cake such that every one feels he or she has gotten as desirable a piece as anyone else (such an allocation of pieces is said to be *envy-free*)? Well, it depends. First, the cake must not be too lumpy. If all of us have our hearts set on getting the cherry in the middle, then it is hopeless unless the cherry can be split up somehow among us. This key property, that any piece can be split up into smaller pieces, corresponds to the idea that our tastes are represented by so-called "atomless" measures, countably additive, etc., etc. In this model, if by a piece one means any measurable subset, then there is a very strong existence result. Not only is there an envy-free allocation, but there is one in which we all believe that everyone, ourselves included, got exactly one *n*th of the cake. Otherwise stated, there is a way of cutting the cake into *n* pieces so that we are all indifferent as to which piece we get. They all look equally delicious. This fact proved by Dubins and Spanier (1961) is a consequence of a celebrated and moderately high-powered theorem of Lyapunov which says that the range of a vector measure is convex.

Now as a practical matter, arbitrary measurable pieces of cake may not be so easy to come by. A more down-to-earth model, therefore, has been treated by Stromquist (1980), where the cake may be taken to be an interval and the pieces are required to be subintervals. Perhaps a loaf of bread is a more apt illustration for this case. Using a fixed-point theorem, it is shown that envy-free allocations will always exist. Fixed-point theorems, however, are notoriously nonconstructive, and Stromquist's result gives no indication of how one might arrive at the desired culinary dissection.

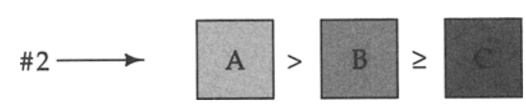
A rather different approach to the problem asks not just for the existence of envy-free allocations but for a procedure, or a *protocol* as we shall call it, which leads to such an allocation. The prototypical example is the procedure for the trivial two-person case where one of us divides the cake into two parts and the other chooses the part he prefers. This method has the obviously desirable property that if either of us ends up feeling he has been gypped, he has only himself to blame. It has long been an open problem to try to devise protocols with this property for the *n*-person case. The best result along these lines is an elegant three-person protocol due to John Selfridge which will now be described. We denote the players by #1, #2, #3.

Three-Person Protocol

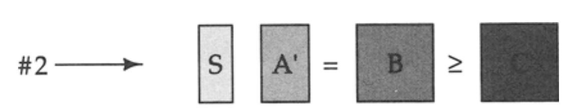
Step I: #1 "trisects" the cake into 3 parts equally acceptable to him.



If #2 and #3 prefer different pieces, we are through. Otherwise, say they both prefer A and #2 prefers A to B which she likes at least as much as C. Thus,



Step II: #2 trims a "sliver" (S) from A leaving A′ so that A′ and B are equally acceptable to her.



Step III: #3 chooses his preferred piece among A′, B, and C.

Case 1. #3 chooses A′. Then #2 chooses B and #1 gets C (no envy so far).

It remains to divide up the sliver which is like the original problem, except that now #1 will not be envious even if #3 gets the whole sliver.

Step IV: #2 trisects S.

Step V: #3 chooses, then #1 chooses, then #2.

Case II. #3 does not choose A′. Then #2 gets A′ and the procedure is as before, except this time #3 trisects and #2 chooses first.

This procedure has a number of nice properties. First, it is economical, requiring at most 5 cuts. Further, like the I-cut–you-choose protocol, it makes minimal assumptions on what the players are able to do; namely, (1) given any piece and an integer k, it is assumed that a player can divide the piece into k subpieces equally acceptable to him, and (2) if a player prefers one piece to another, she can trim off part of the first piece in such a way that what is left and the second piece are equally acceptable to her. Finally, preferences are required to be only *weakly additive*, meaning that if A, B, and X, Y are disjoint pieces and a player prefers A to X and B to Y, then he also prefers $A \cup B$ to $X \cup Y$.

Up to now, no protocol satisfying the desired conditions is known, even for the case of four players. However, recent work by Steven Brams and Alan Taylor seems to indicate that some progress is being made. The authors present what they call a finite algorithm for arriving at an envy-free allocation. Their procedure, however, is quite complicated and seems to require that players be able to measure numerically the value of any piece of cake. Further, even for the four-player case, there is no *a priori* upper bound on the number of cuts which may be required. Thus, for example, if the value of piece A to some player is greater than that of piece B by one part in a million, then it may require a million cuts to arrive at the desired allocation using the proposed algorithm. One might hope that procedures of simplicity comparable to that of the Selfridge protocol could be devised for the general case. As this is being written, however, the new work is still at the early (preprint) stages, and perhaps substantial simplifications will be found in the course of time.

Dividing a Pie: an Unsolved Problem

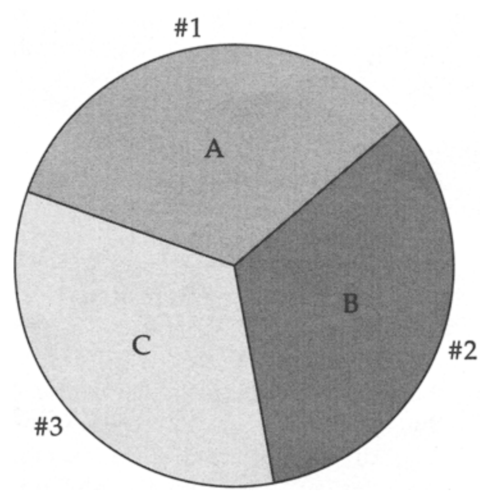
An allocation may be envy-free but have other undesirable properties. As an example, suppose you and I are to divide a loaf of bread which again we will take to be an interval, and suppose the loaf is symmetric about its midpoint in both of our measures. Then if we divide it in two at the midpoint, we have a Spanier–Dubins allocation in which we both agree that each of us got exactly half the cake. Suppose, however, that I like crust, so that I particularly want to get the two ends of the loaf, whereas you prefer not to have these parts. Then each of us would be strictly better off if we trisected the loaf in some way and you took the middle part while I took the two end intervals.

In general, we will say an allocation is *dominated* if there is another allocation which gives all players pieces they strictly prefer. Obviously, it would be desirable for the final allocation to be undominated as well as envy-free. A general question, then, is whether in a given model it is always possible to satisfy both of these conditions. In this connection, recall that in the Stromquist formulation all pieces were required to be subintervals of an interval (unlike the earlier example in which my allocation was the union of two disjoint intervals). For these Stromquist allocations, we have, in fact,

THEOREM. *An envy-free Stromquist allocation is automatically undominated.*

Proof. Let P be an envy-free partition of the interval into n subintervals and let Q be any other such n-partition. Now, if P and Q are distinct n-partitions of an interval, then there must be some interval I of P which strictly contains some interval J of Q (think about it for a minute). But then whoever gets J in the allocation Q will not be strictly better off than she was under P, for she likes I at least as well as J and she likes the piece she got under P at least as well as I since P was envy-free. Q.E.D.

Which brings us to the problem of the pie. Suppose a pie is to be divided among three people and the pieces are required to be traditional pie portions, namely, *sectors.*

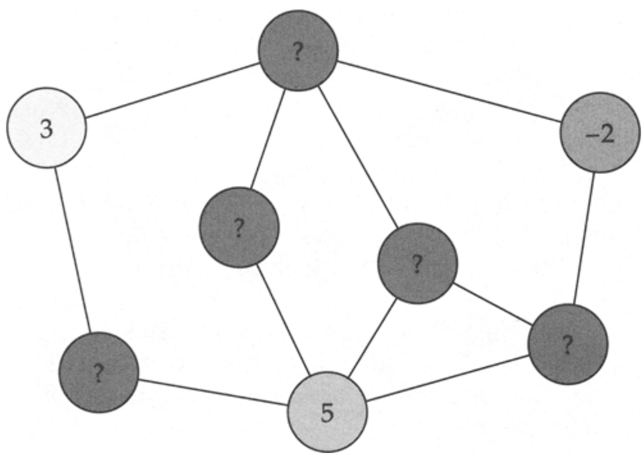


Does there necessarily always exist an allocation which is both envy-free and undominated?

Addendum on the Variational Method

There is, of course, an inexhaustible supply of problems that can be solved by variational methods, the subject of the last issue's column, but I missed a lovely two-examples-in-one case which was suggested to me by Clifford Gardner. It has the special virtue that it provides a simple but elegant application of calculus and would fit in at around the third week of a traditional freshman course. (The NSF has been spending millions in recent years on trying to improve the teaching of calculus, so I am pleased to be able here to contribute my own two cents' worth.)

Consider the *discrete heat-flow problem.* Given a graph like the one shown below, where some of the nodes are held at fixed temperatures (3, 5 and −2); the laws of heat flow require that the (steady-state) temperature of every remaining node shall satisfy the *mean value property*, namely, that its temperature shall be the average of the temperatures of the nodes to which it is connected.



Question

How do we know that such a set of temperatures will always exist, and if so are they unique?

(Of course, this is a problem of solving a system of linear equations, but our students will not get to this until their sophomore year.)

Answer

Existence: Let t_i be the temperature at node i, and consider the function $f(t_1, \ldots, t_n) = \Sigma (t_i - t_j)^2$, summed over all pairs of neighboring nodes (this is the *thermal energy* of the system); choose values of the t's which minimize this function. To see that these values satisfy the desired condition, note that if the temperatures at all nodes except t_k are held fixed at the minimizing values, then t_k must minimize f as a function of one variable, and setting the derivative with respect to t_k equal to zero gives the result.

Ah, you say, but how do we know that the minimum exists? My answer is that mathematics got along for two thousand years without worrying about such questions and there is no reason to inflict them on freshmen. For those who want to be mathematics majors, there will be time enough when they get to be juniors to force them, kicking and screaming in some cases, to worry about these matters.

Uniqueness (by the maximum principle!): Suppose there were two sets of temperatures satisfying the mean value property. Then so would their difference, and the difference temperatures at the fixed nodes would be zero. Now consider a node where this difference temperature is a maximum. Then by the mean value property, all its neighbors must also be at this temperature, and likewise all its neighbors' neighbors, so eventually we will reach one of the fixed-temperature nodes (we assume the graph is connected); so the maximum is zero. Likewise the minimum.

Q.E.D.

The Patron Saint of Mathematics

R. J. Duffin

God created the world and the integers, all in seven days. He then ordered two of his biotechnicians, James and Francis, to construct a genetic code for the fractional numbers. Moreover, they were to give special prominence to His favorite number, π.

Because the world was created in seven days they chose a code with seven bases instead of the biochemical foursome A,G,T,C. The bases are called 1,2,3,4,5,6,7. For any number x they devised an algorithm to give the sequence of bases X_1, X_2, $X_3, \ldots, X_k$ in a code called *signature seven:*

$$X_n = \text{Next}[7\ \text{Frac}(nx)] \quad \text{for } n = 1, 2, 3, \text{etc.}$$

Here Frac(y) gives the fractional part of a number y and Next[y] gives the smallest integer at least as large as y.

When finished, they presented their handiwork at a Seminar lecture titled "Π equals 1234567." They found signature seven of π:

Π_n = Next[7 Frac($n\pi$)] for seventy signature places to be

Π = 12345671234567 . . . 1234567.

This is 10 perfect copies of π. The Boss was so pleased that He promoted James and Francis to Archangels on the spot.

At the Seminar, Satan gave a devilish smile because they had not gone far enough to detect a flaw which he had secretly introduced. The flaw is the numerophage virus, *irrationality*. This virus is recessive; often lying dormant waiting to strike.

Irrationality is a serious disease of the System of the World. Mathematicians proclaim that they can cure this malady. At least they profit by treating symptoms. So Satan is their patron and benefactor.

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213–3890
USA