Introduction to Linear Algebra

(Reference: Kreyzig)

① $a \rightarrow \boxed{L} \rightarrow L\{a\}$

$b \rightarrow \boxed{L} \rightarrow L\{b\}$ mapping (transformation)

∴ What is a $\boxed{\text{linear operator}}$? (2 conditions)

(1)

(2)

• Is matrix multiplication a linear transformation?

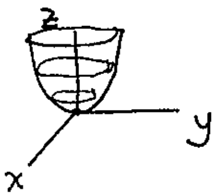
$$\begin{bmatrix} r \\ s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \rightarrow$$

$\mathbb{R}^3$ $\qquad$ $\mathbb{R}^2$

• Is this a linear transformation?

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

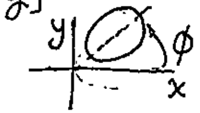
$\begin{bmatrix} x \\ y \end{bmatrix}$ $\qquad z = x^2 + y^2$



• Is this a linear transformation? (orthonormal bases related by rotation)

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \quad \begin{bmatrix} X \\ Y \end{bmatrix}$$



② Motivation: Transforming from one space to another

Matrix Multiplication

$$\vec{w} \longrightarrow \boxed{B} \xrightarrow{\vec{x}} \boxed{A} \longrightarrow \vec{y}$$

$\mathbb{R}^{\square}$ $\qquad$ $\mathbb{R}^{\square}$ $\qquad$ $\mathbb{R}^{\square}$

$$\begin{cases} x_1 = w_1 + 2w_2 \\ x_2 = 3w_1 + 4w_2 \\ x_3 = 2w_1 + w_2 \end{cases} \qquad \begin{cases} y_1 = x_1 + 2x_2 + 3x_3 \\ y_2 = 2x_1 + x_2 + x_3 \end{cases}$$

$\vec{y} = \ ? \ \vec{w}$ (Write a linear transformation that maps $\vec{w}$ to $\vec{y}$)

③ Matrix Multiplication

Sigma Notation

Dot Product $\quad \vec{a}^T \vec{b} = [a_1 \ a_2 \cdots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$

$$\underset{(m\times r)(r\times n)}{C = AB} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

$C_{ij} =$

④ Matrix multiplication Properties

[T/F] $(kA)B = k(AB)$

[T/F] $(AB)C = A(BC)$

[T/F] $(A+B)C = AC + BC$

[T/F] $C(A+B) = CA + CB$

[T/F] $AB = BA$

$\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} =$, $\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix} =$

[T/F] $AB = 0 \Rightarrow A = 0$ or $B = 0$

$BA = 0$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} =$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} =$

[T/F] $AC = AD \Rightarrow C = D$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} =$, $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} =$

[T/F] $(A+B)(A+B) = A^2 + 2AB + B^2$ why

[T/F] $AB = 0$ and $|B| \neq 0 \Rightarrow A = 0$ why

[T/F] $AC = AD$ and $|A| \neq 0 \Rightarrow C = D$

ⓐ [T/F] A is singular $\Rightarrow A^T$ is singular why

[T/F] A is singular $\Rightarrow BA$ is singular $\Rightarrow AB$ is singular

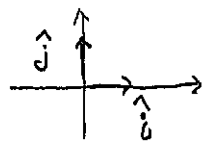
⑤ $\boxed{\text{Vector Space}}$ $V =$ set of all vectors s.t.

$\vec{a} \in V$
$\vec{b} \in V$ $\Rightarrow$

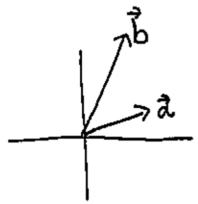
(the two operations of ______ and ______ are defined)

$\boxed{\text{basis}} =$

dimension $\boxed{\dim V} =$

- 

$\mathbb{R}^2$ is a vector space
Basis { }
All vectors can be written in terms of
$\dim(\mathbb{R}^2) =$

- 

• does $\{\vec{a}, \vec{b}\}$ also form a basis for $\mathbb{R}^2$

- Vector Space $\mathbb{R}^n$ has dimension ____
why? one basis is { }

Can you see each cannot be written in terms of the others?

why

ⓐ $(AB)^T =$

$(ABCD)^{-1} =$ why

6. $\boxed{\text{Linear Independence}}$

A set of vectors are linearly independent of each other if ...

- $\vec{a} = [1 \quad 3 \quad 1]$ $\rightarrow$ $\vec{c} = 2\vec{a} + 3\vec{b}$
  $\vec{b} = [1 \quad -2 \quad 3]$ $\nearrow$ $=$

  Are $\vec{a}, \vec{b}, \vec{c}$ linearly independent?

  $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -2 & 3 \\ 5 & 0 & 11 \end{array}\right]$ In Gauss elimination, a row of 0's means the 3rd row...

- Linear combination of $\vec{a}, \vec{b}, \vec{c}$ ...

- $\boxed{k_1\vec{a} + k_2\vec{b} + k_3\vec{c} + \ldots = \vec{0}}$
  $\vec{a}, \vec{b}, \vec{c}, \ldots$ are linearly independent iff ______

- $\vec{a} = [3 \quad 0 \quad 2 \quad 2]$
  $\vec{b} = [-6 \quad 42 \quad 24 \quad 54]$
  $\vec{c} = [21 \quad -21 \quad 0 \quad -15]$

  Are $\vec{a}, \vec{b}, \vec{c}$ independent? If not, write $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$

7. $\boxed{\text{rank}}$ $\underset{\text{of matrix}}{A}$ = $\underset{\text{could prove}}{=}$

- $\text{rank}(A^T) =$

8. After Gauss $\overset{\text{Jordan}}{\vee}$ elimination, the only possibilities are

(A = coefficient matrix, $\tilde{A}$ = augmented matrix)
m equations, n unknowns

$m < n$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$m = n$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$m > n$
some info redundant

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\text{rank}\,\tilde{A} < \text{rank}\,A \Rightarrow$

$\text{rank}\,\tilde{A} > \text{rank}\,A \Rightarrow$

$\text{rank}\,\tilde{A} = \text{rank}\,A \Rightarrow$

and $= n$ unknowns $\Rightarrow$

9. To find A^{-1}

$[A \mid I] \xrightarrow[\text{Gauss Jordan}]{} [I \mid A^{-1}]$ Explain the process (hamsters, gerbils)

⑨ Homogeneous System

$A\vec{x} = $ ___
$n\times n$

- nontrivial ($\vec{x} \neq \vec{0}$) exists only if ________
- Can there be only one solution to an $m\times n$ system? Why?
- $$\begin{array}{c} \begin{array}{ccccc|c} x & y & z & s & t & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \leftarrow \text{Solve } \begin{cases} x+2y+3z+4s+5t=0 \\ y+z+s=0 \end{cases}$$

$A\vec{x} = \vec{0}$

write as linear combination of basis vectors

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} =$$

So any one of the infinitely many solutions $\vec{x}$ is a linear combination of the basis vectors.

$\{\vec{x}\}$ = N = ________ $\Big\{ \qquad \Big\}$
soln set to $A\vec{x}=\vec{0}$

nullity = ______
dim(N)

- $\vec{x}_1$ is a solution
  $\vec{x}_2$ "
  $\Rightarrow \alpha\vec{x}_1 + \beta\vec{x}_2$ is a solution since ...
- Thm $\boxed{\qquad\qquad}$

⑩ Properties of determinants (Staple proofs)

*(1) Expanding along any row or column gives the same result

(2) $\underset{n\times n}{|A|} = \Big| \qquad \Big| =$ sigma notation for i^{th} column expansion

(3) Computationally,
- Gauss & back-substitution is best (Jordan is slower than)
- $n\times n$ determinant takes $O(\quad)$ multiplications Why?
- If one multiplication takes 10^{-9} seconds,

n	10	15	20	25
Time				

*(h) $|AB| =$

(a) interchange rows $\Rightarrow$ det $\times$ ____
(b) $k\times$row $\Rightarrow$
det(kA) $\Rightarrow$
(c) det(A^T) =
(d) 0 row or column $\Rightarrow$
*(e) proportional rows or columns $\Rightarrow$
(f) triangular's det $\begin{vmatrix} \lambda_1 & \cdots \\ 0 & \lambda_2 \ddots \lambda_n \end{vmatrix} \Rightarrow$
*(g) combining rows or columns $\Rightarrow$

*(i) $n\times n$ system
$\boxed{\begin{array}{l} A\vec{x}=\vec{b} \text{ has unique soln} \\ \Leftrightarrow A^{-1} \text{ exists} \\ \Leftrightarrow |A| \neq 0 \end{array}}$

⑪ Prove
$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}$$
etc.

⑫ Prove Cramer's Rule