- 8.2 Start familiarizing with the polar graph families.
- 8.2 (Pg. 594) # 1~6, 9, 11~14, (15~34 only need to sketch using the pattern you memorize);
  553 3~8 11 13~16 17~40
  23 (28), 27
  (30, 32, 33, 37, 39, 41 actually plot out the points); 43~46, 48, 49.
  30 37 33 43 46 47 49~52 54 56
- Ch. 8 Review # 1~11 odd, 13~24 every, (25,) 27.
  P572 2~12 even 13~24 every pattern 26, 27 (CALC)
  20 22 conics

Quiz

- 8.2 pattern
- Bonus: For a reciprocal spiral r=a/theta, show that y=a is an asymptote by converting a limit in x and y into a limit with theta. Hint: lim as theta --> 0 of sin(theta)/theta = 1.

Polar Bonus

B1) Prove using standard form that $(z_1/z_2)^* = z_1^*/z_2^*$.

B2) Prove using phasors that $(z_1z_2)^* = z_1^*z_2^*$ and $(z_1/z_2)^* = z_1^*/z_2^*$.

B3) Prove the generalization of the triangle inequality using induction (or just your logic).

B4) Prove the last line on pg. 599 using polar (not phasor) form. "The proof of the division formula is left as an exercise."

B5) Prove arg{1/z} = -arg{z} using standard form and using phasors.

B6) Prove that a circle with diameter on y-axis has r=2asin(theta). B7) Prove why a triangle inscribed in a circle with diameter as a triangle sidemust be a right triangle.

CW

cw 13/7 11/8 16/11 18/13 19/15 21/17

- 8.3 DeMoivre (P. 603) # 7~18, 22, 23, 24, 29, 30, 32, 34, 37, 39, 48, 50, 52, | 53, 59, 63,
  8.3 (P.562) 12~22 26 28 27 33 34 35 37 41 44 52 53 56 | 58 63 68
  67, 75, 76, 78, 83, 86, 89, 90. Bonus # 94, 95, 96.
  72 79 80 82 85 87 90 94 93 98 99 100

8.4

- 10.7 (Parametric Pg. 807) # 9, 10, 11, 12, 20, 21, 28, 29, 30, 35, 40, 45, 47~50, 62. Bonus # 53, 58, 61.
  8.4 (570) 12 11 14 13 21 24 34 35 36 41 45 51 53~56 why 59 64 67

- Parametric Extension (P. 575 in 6th)

CW

• 8.4 (Vector P. 615) #	3,	5,	6,	8,	9,	16,	18,	21,	24,	25,	29,	30,	34,	35,	37,	41,	43,	48,
9.1 (P.587)	5	8	7	10	11	17	31	36	38	40	44	43	48	50	51	55	59	64
	51,	52,	55,	56,	57.													
Quiz:	67	68	71	72	73													

- 8.3 Polar, DeMoivre, nth roots
- 8.4 Vectors
- 8.5 Basic dot and cross products

														Demo		
• 8.5 (Dot Product P. 624) #	2,	3,	5,	12,	13,	15,	17,	18,	19,	20,	21,	24,	25,	27,	30,	31,
9.2 (P596)	6	8	9	18	20 like	22	24	23	25	26	28	30	32	34	36	38
	34,	35,	36,	38,	43,	46,	47.									
	40	42	41	44	49	52	53									

- 9.3 (3D) # 6, 10, 12, 22

8.1 (P.586)

Rose Shortcut

$r(\theta) \to$ tangents $r_{max} \to$ petal tips odd/even?

#7, 9, 11, 13 ~ 23 odd, 27, 31, 35, 39, 41, 42 43, 46 ~~49~~ 50, 51, 52, 53, 57, 59, 61

8.2 (P.594)

#1~6, 9, 11, 12, 13, 14, 15 ~ 34
Just Sketch by pattern

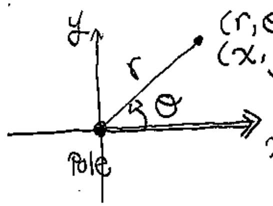
Actually graph: 30, 32, 33,
37, 39, 41 48, 49
Bonus 43~46

8.3 (P.603)
7 8 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 29, 30
32, 34, 37, 39, 48, 50, 52, 53, 59, 63, 66, 67, 75, 76, 78,
83, 86, 89, 90, *94, 95, 96

8.4 (P.615)
3, 5, 6, 8, 9, 16, 18, 21, 24, 25, 29, 30, 34, 35, 37
41, 43, 48, 51, 52, 55, 56, 57

8.5 (P.624)
2, 3, 5, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25, 27, 30, 31, 34, 35, 36, 38, 43,
46, 47

8.1 Polar Coordinates



$$\boxed{\begin{array}{c} r = \sqrt{x^2+y^2} \\ \alpha = \tan^{-1}\left(\frac{y}{x}\right)\end{array}}$$

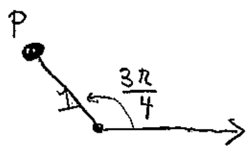
reference angle

only in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

★Check if not in QI, IV

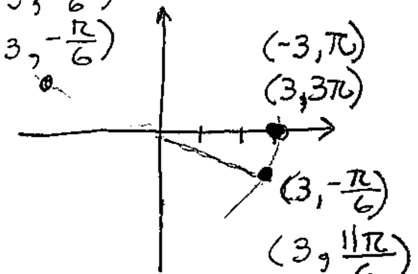
$$\boxed{\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \end{array}}$$

ex1 $\left(1, \frac{3\pi}{4}\right)$



$\left(-3, \frac{11\pi}{6}\right)$

$\left(-3, -\frac{\pi}{6}\right)$



not unique

$(r, \theta) = (-r, \theta + \pi) = (-r, \theta + \pi + 2n\pi)$

$= (r, \theta + n2\pi)$

ex3 $\left(4, \frac{2\pi}{3}\right) \Rightarrow (x, y) = \boxed{(-2, 2\sqrt{3})}$

$x = 4\cos\frac{2\pi}{3} = -4 \cdot \frac{1}{2} = -2$

$y = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

60°

ex $(-8, 1) \to (r, \theta)$

$r = \sqrt{1^2+8^2} = \sqrt{65}$

$\tan\gamma = \frac{1}{8} \Rightarrow 7°$

$\theta = 180° - \gamma = 173°$

ex4 $(-2, -2) \longrightarrow (r, \theta) = \boxed{\left(2\sqrt{2}, \frac{5\pi}{4}\right)}$

$r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\left|\frac{2}{2}\right|\right) = \tan^{-1} 1 = 45°$

Polar Eqn

ex5 $x^2 = 4y$ in polar

$(r\cos\theta)^2 = 4r\sin\theta$

$r^{\cancel{2}}\cos^2\theta = 4\cancel{r}\sin\theta \quad r \neq 0$

$r = \frac{4\sin\theta}{\cos^2\theta}$

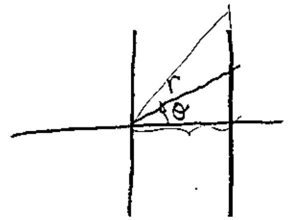
$$\boxed{r = 4\sec\theta\tan\theta}$$

4) Rose
try symm
$r = 2\cos(2\theta)$

ex6 a) $r = 5\sec\theta$

$r\cos\theta = 5$

$x = 5$



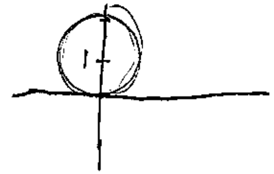
b) $r = 2\sin\theta$

$r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$



x: $(r, -\theta)$
$(r, 2\pi - \theta)$ ✓

y: $(-r, -\theta)$ ×!
$(r, \pi - \theta)$ ✓

orig: $(r, \pi + \theta)$ ✓
$(-r, \theta)$ ×!

ex: rose
limacon
lemniscate
Graphing CALC ex7

c) $r = 2 + 2\cos\theta$

$r^2 = 2r + 2r\cos\theta$

$x^2 + y^2 = 2r + 2x$

$(x^2 + y^2 - 2x)^2 = (2r)^2$

$$\boxed{(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)}$$

8.2
1) Line, circle, line thru origin
ex1 ex2

2) Limaçon. Show θ/r idea
Give examples Pattern $3 - 3\cos\theta$ $5 + 5\sin\theta$

$2 + 1\cos\theta$ $3 - 1\sin\theta$

$3 + 5\sin\theta$ $2 + 7\cos\theta$

8.2 See Calc BC notes
Polar eqns pg. & pg 2~5

3) Symmetry
ex6 $r = 1 + 2\cos\theta$

y-symm: $(r, -\theta)$ × $(r, \pi - \theta)$ ×
x-symm: $(r, -\theta)$ ✓

orig: (r, θ) × $(r, \pi + \theta)$ ×

8.3 Complex numbers

reference: Kreyzig, Engineering Math (12.1, 12.2, 12.6)

Complex number $z = x + iy$

real part $x = \mathrm{Re}\, z$

imaginary part $y = \mathrm{Im}\, z$

real number: $3 = 3 + 0i$

$(i = 0 + 1i)$

addition $\boxed{z_1 \pm z_2} \triangleq (x_1 \pm x_2) + i(y_1 \pm y_2)$

$(x_1, y_1) + (x_2, y_2)$

$\boxed{z_1 z_2} \overset{\text{definition}}{=} x_1 x_2 - y_1 y_2 + i(x_1 y_1 + x_2 y_2)$

$(x_1 + iy_1)(x_2 + iy_2)$

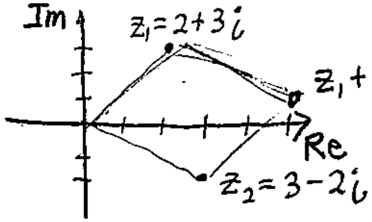
$\boxed{i^2 = -1}$ $\because (0 + 1i)(0 + 1i) = -1$

$$\boxed{\frac{z_1}{z_2}} = \frac{8+3i}{9-2i} \cdot \frac{9+2i}{9+2i} = \frac{72 - 6 + (27+16)i}{81+4}$$

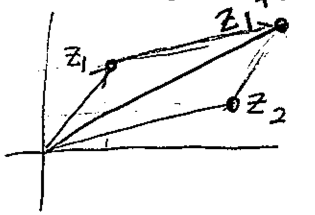
$$= \frac{66}{85} + \frac{43}{85} i$$

Complex plane

ex1



walking along $z_1 + z_2$

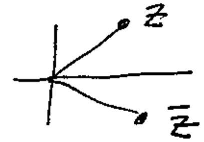


ex2 Graph

a) $S = \{a + bi \mid a \geq 0\}$

b) $T = \{a + bi \mid a < 1,\ b \geq 0\}$

Complex Conjugate $z = x + iy$, $\bar{z} = x - iy$



$$\begin{cases} \mathrm{Re}\, z = \frac{1}{2}(z + \bar{z}) \\ \mathrm{Im}\, z = \frac{1}{2i}(z - \bar{z}) \\ z\bar{z} = x^2 + y^2 \quad \because (x+iy)(x-iy) = x^2 + y^2 \end{cases}$$

$$\begin{cases} \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \quad ① \\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad ② \end{cases}$$

verify ① $\bar{z}_1 \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1)$

$\overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)}$

BONUS ② $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{x_1 - iy_1}{x_2 - iy_2} \cdot \dfrac{x_2 + iy_2}{x_2 + iy_2} = \dfrac{x_1 x_2 + y_1 y_2 + i(x_1 y_2 - x_2 y_1)}{x_2^2 + y_2^2}$

$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\dfrac{x_1 + iy_1}{x_2 + iy_2} \cdot \dfrac{x_2 - iy_2}{x_2 - iy_2}} = \overline{\dfrac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}}$

$\boxed{\text{OR}}$ $\underbrace{r_1 e^{-i\theta_1}}_{\bar{z}_1} \underbrace{r_2 e^{-i\theta_2}}_{\bar{z}_2} = r_1 r_2 e^{-i(\theta_1 + \theta_2)} = \overline{z_1 z_2}$

$\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}} = \dfrac{r_1}{r_2} e^{-i(\theta_1 - \theta_2)} = \overline{\left(\frac{z_1}{z_2}\right)}$

ex $z_1 = 4+3i$

$z_2 = 2+5i$

$$\operatorname{Im} z_1 = 3 = \frac{1}{2i}(z_1 - \bar{z}_1) = \frac{1}{2i}(\cancel{4}+3i-\cancel{(4}-3i)) = \frac{6i}{2i} = 3$$

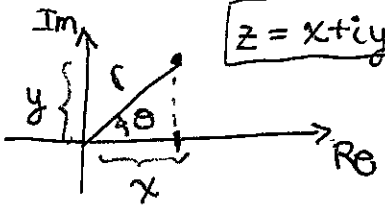
$$\overline{z_1 z_2} \stackrel{?}{=} \bar{z}_1\,\bar{z}_2 = (4-3i)(2-5i)$$

$$= (8+15) + i(-20-6) = -7-26i$$

$$\overline{(8-15)+(20+6)i} = -7-26i \quad \checkmark$$

Polar Form

Phasor



$$\boxed{z = x+iy = r(\cos\theta + i\sin\theta)} = re^{i\theta}$$

$z = x+iy$

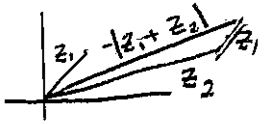
Absolute Value (magnitude), modulus, norm $\quad |z| = r = \boxed{\sqrt{x^2+y^2}} = \sqrt{z\bar{z}}$

Argument $\quad \boxed{\theta = \arg z} = \tan^{-1}\left(\frac{y}{x}\right)$

$+\ 2n\pi \qquad$ reference

Triangle Inequality

$$|z_1+z_2| \le |z_1| + |z_2|$$

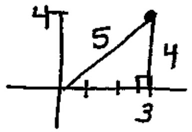


By induction $\quad |z_1+z_2+\dots+z_n| \le |z_1|+|z_2|+\dots+|z_n|$

Calculate the modulus

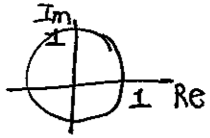
ex3 $z_1 = 3+4i$ $\qquad z_2 = 8-5i$

$|z_1| = \sqrt{9+16} = \sqrt{25} = 5 \qquad |z_2| = \sqrt{64+25} = \sqrt{89}$

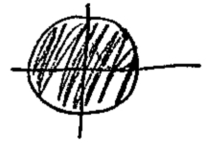


ex4 Graph

a) $C = \{z \mid |z| = 1\}$



b) $D = \{z \mid |z| \le 1\}$

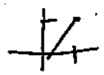


ex5 write in polar form & sketch

a) $1+i = \underline{r}\,(\cos\underline{\theta} + i\sin\theta)$ Show r & θ

Standard form

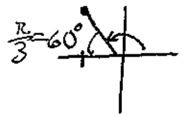
$= \boxed{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$



$r = \sqrt{1+1}$

$\theta = 45° = \tan^{-1}\left(\frac{1}{1}\right)$ $\qquad$ another $\arg = \frac{\pi}{4} + 2\pi$

b) $-1+\sqrt{3}\,i = \boxed{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$

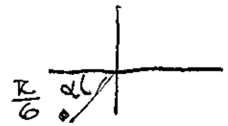


$r = \sqrt{1+3} = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

$\theta = \frac{2\pi}{3}$

c) $-4\sqrt{3}-4i = \boxed{8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)}$

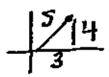


$r = 4(\sqrt{3+1}) = 8$

$\alpha = \tan^{-1}\left(\frac{4}{4\sqrt{3}}\right) = 30°$

$\theta = \frac{7\pi}{6}$

d) $3+4i = \boxed{5\left(\cos\left(\tan^{-1}\frac{4}{3}\right) + i\sin\left(\tan^{-1}\frac{4}{3}\right)\right)}$

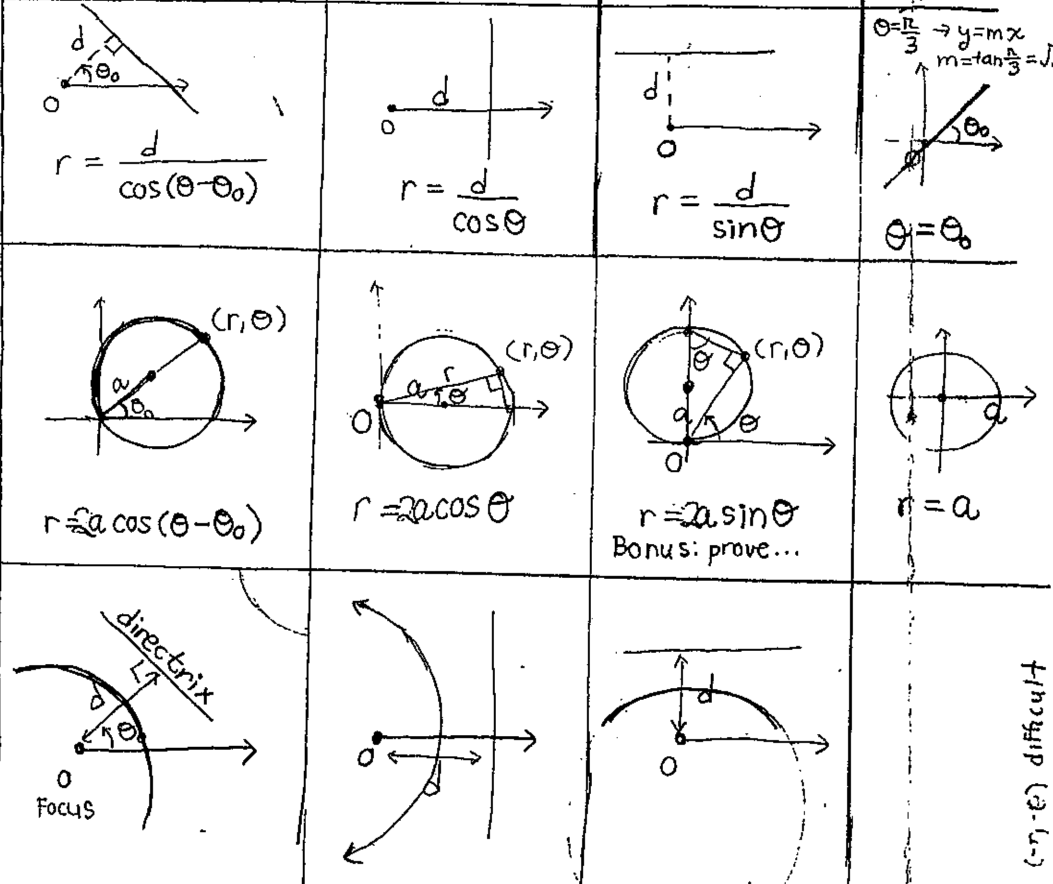


(Anton 11.1)

12.6 Polar Equations

Symmetry: $(r,-\theta)$, $(r,\pi-\theta)$, $(-r,-\theta)$, $(r, 2\pi-\theta)$, $(-r,\theta)$, $(r,\theta+\pi)$

	General Case	$\theta_0 = 0$	$\theta_0 = \pi/2$	Special
Line	$r = \dfrac{d}{\cos(\theta-\theta_0)}$	$r = \dfrac{d}{\cos\theta}$	$r = \dfrac{d}{\sin\theta}$	$\theta = \frac{\pi}{3} \to y = mx$, $m = \tan\frac{\pi}{3} = \sqrt{3}$; $\theta = \theta_0$
Circle	$r = 2a\cos(\theta-\theta_0)$	$r = 2a\cos\theta$	$r = 2a\sin\theta$ Bonus: prove...	$r = a$
Conics: Ellipse $(0<e<1)$, Parabola $(e=1)$, Hyperbola $(e>1)$	$r = \dfrac{ed}{1+e\cos(\theta-\theta_0)}$	$r = \dfrac{ed}{1+e\cos\theta}$	$r = \dfrac{ed}{1+e\sin\theta}$	

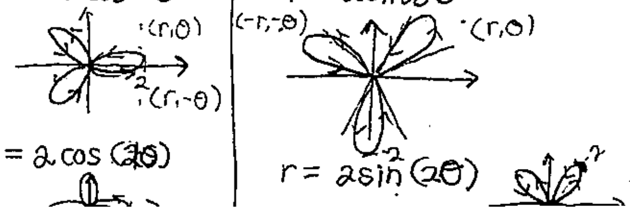


Keplar's Laws of Planetary Motion

③ $\boxed{\text{Rose}}$ $r = a\cos n\theta$ ← symm about x-axis; $r = a\sin n\theta$ ← symm about y-axis

n odd: leaves — $r = 2\cos 3\theta$, $r = 2\sin 3\theta$

n even: 2n leaves — try symm about — $r = 2\cos(2\theta)$, $r = 2\sin(2\theta)$



symm. about y $(-r,-\theta)$ (about x $(r, \pi-\theta)$)

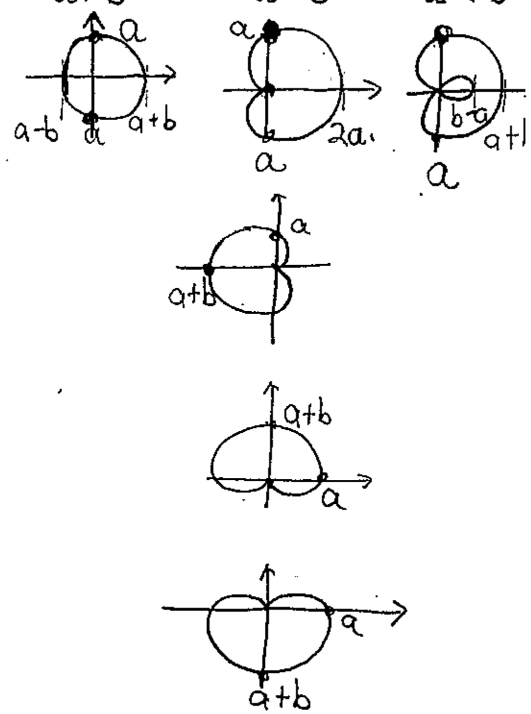
12.7 Graphs of polar equations

★ It helps to find the tangent lines by solving $r(\theta) = 0$

① $\boxed{\text{Limaçon}}$: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$

$\boxed{\text{cardioid}}$: when $a = b$

	Limaçons	cardioid: $a>b$	$a=b$	$a<b$
$(r,-\theta)$ x-axis symm	$r = a+b\cos\theta$			
	$r = a-b\cos\theta$			
$(-r,-\theta)$ difficult; $(r,\pi-\theta)$, $\sin(\pi-\theta) = -\sin(\theta-\pi) = \sin\theta$; symm y-axis	$r = a+b\sin\theta$			
	$r = a-b\sin\theta$			

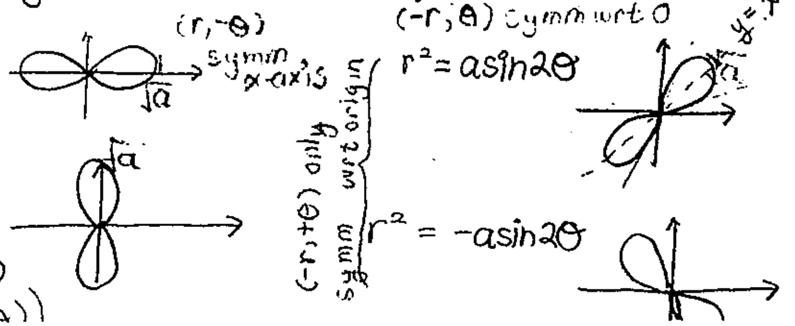


② $\boxed{\text{Lemniscate}}$ (Figure 8) $r^2 = \pm a\cos 2\theta$, $r^2 = \pm a\sin 2\theta$

$(r,-\theta)$; $(-r,\theta)$ symm wrt O

symm x-axis $(r,-\theta)$, symm y-axis $(-r,-\theta)$, symm origin $(-r,\theta)$: $r^2 = a\cos 2\theta$ — symm x-axis; $r^2 = -a\cos 2\theta$

$(-r,\theta)$ only symm wrt origin: $r^2 = a\sin 2\theta$; $r^2 = -a\sin 2\theta$



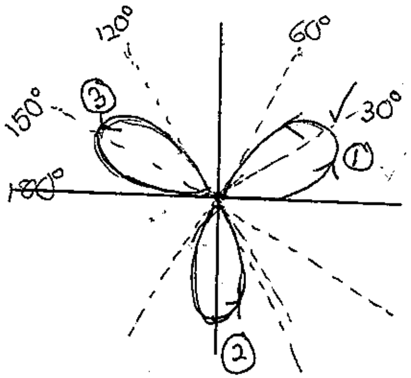
Sketch

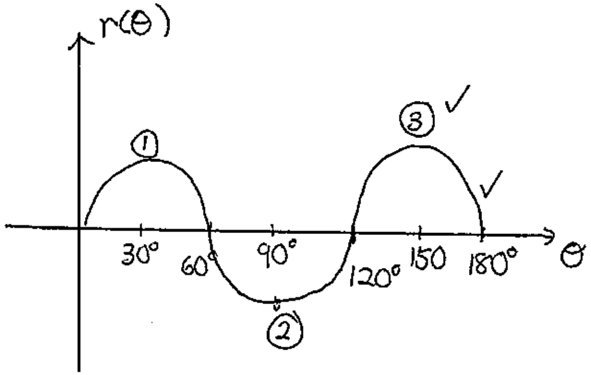
$\boxed{ex}$ $y = 2\sin(3\theta)$ $\quad \frac{360^\circ}{3} = 120^\circ$

demo

Classwork

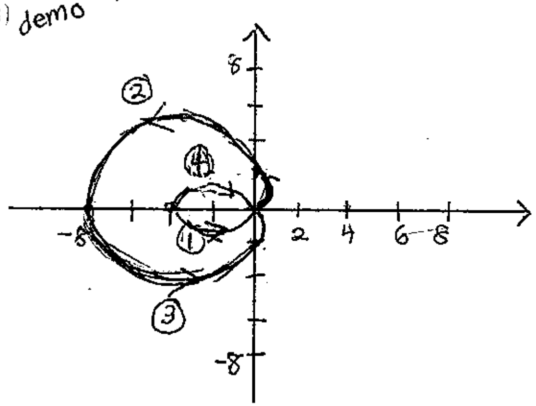
$3\times6 \uparrow$ 25 pts

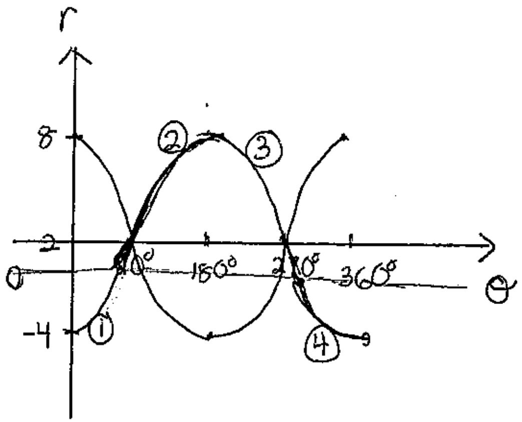




$\boxed{ex}$ $r = 2+6\cos\theta$

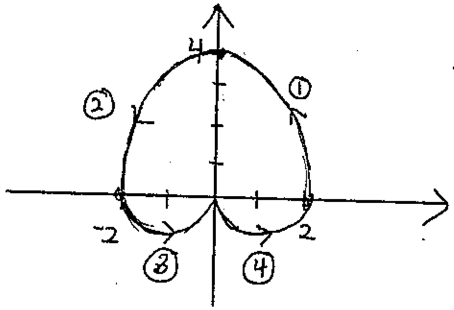
(1) demo

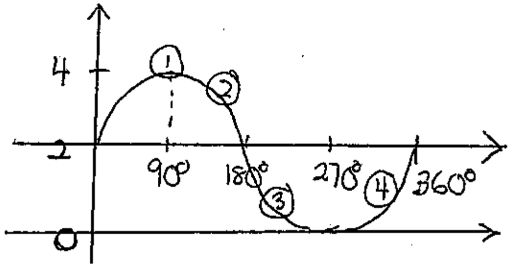




$\boxed{ex}$ $r = 2+2\sin\theta$

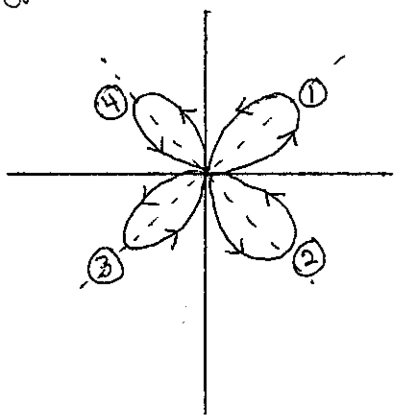
CW
1) graph
2) 45° etc
3) number & explain

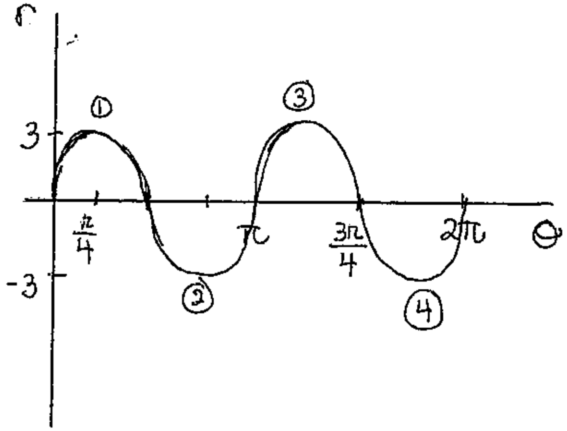




$\boxed{ex}$ $y = 3\sin(2\theta)$

CW





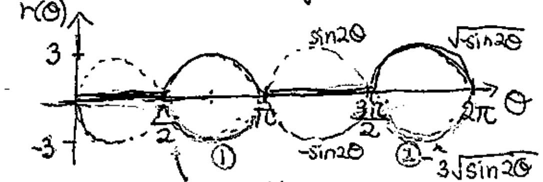
$\boxed{ex}$ $r^2 = 4\cos 2\theta$

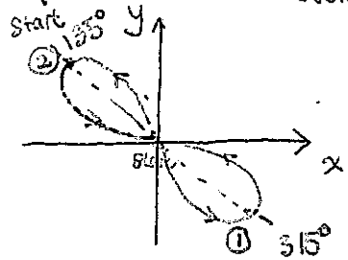
demo

$\boxed{ex}$ $r^2 = -9\sin 2\theta$

CW

$r = \pm 3\sqrt{-\sin 2\theta}$

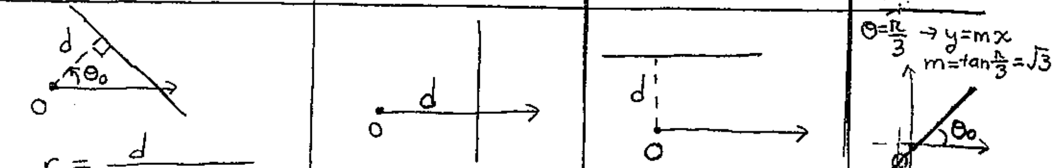


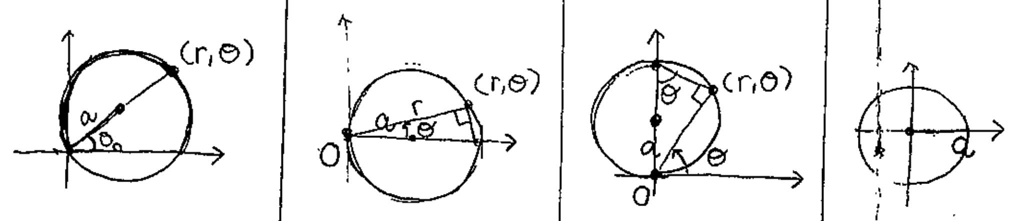


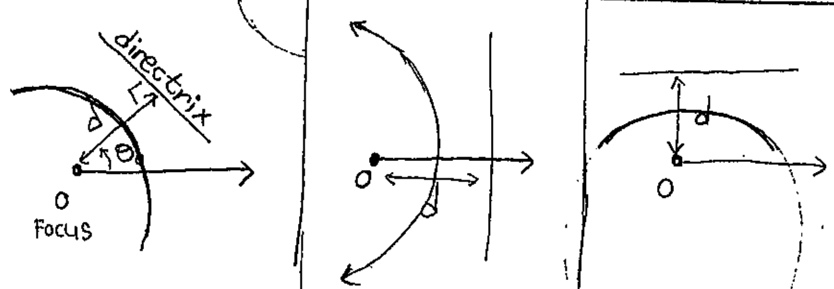
(Anton 11.1) **12.6 Polar Equations**

symmetry $(r,-\theta)$ $(r,\pi-\theta)$ $(-r,-\theta)$ (r,θ) $(-r,\theta)$ $(r,\theta+\pi)$ $(r,2\pi-\theta)$

	General Case	$\theta_0=0$	$\theta_0=\pi/2$	Special
Line	$r=\dfrac{d}{\cos(\theta-\theta_0)}$	$r=\dfrac{d}{\cos\theta}$	$r=\dfrac{d}{\sin\theta}$	$\theta=\frac{\pi}{3}\to y=mx$, $m=\tan\frac{\pi}{3}=\sqrt{3}$; $\theta=\theta_0$
Circle	$r=2a\cos(\theta-\theta_0)$	$r=2a\cos\theta$	$r=2a\sin\theta$ Bonus: prove...	$r=a$
Conics: Ellipse $(0<e<1)$, Parabola $(e=1)$, Hyperbola $(e>1)$	$r=\dfrac{ed}{1+e\cos(\theta-\theta_0)}$	$r=\dfrac{ed}{1+e\cos\theta}$	$r=\dfrac{ed}{1+e\sin\theta}$	



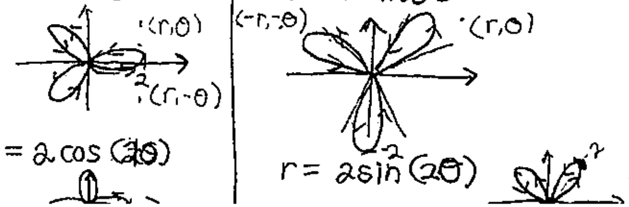




Keplar's Laws of Planetary Motion

③ $\boxed{\text{Rose}}$

$r=a\cos n\theta$ (symm about x-axis)	$r=a\sin n\theta$ (symm about y-axis)
n odd: leaves $r=2\cos 3\theta$	$r=2\sin 3\theta$
n even: 2n leaves, try symm about $r=2\cos(2\theta)$	$r=2\sin(2\theta)$



symm. about y $(-r,-\theta)$ (about x $(r,\pi-\theta)$)

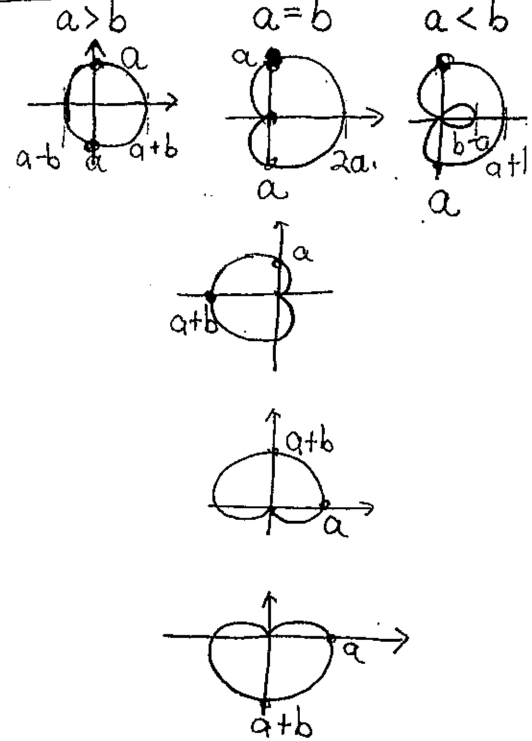
12.7 Graphs of polar equations

* It helps to find the tangent lines by solving $r(\theta)=0$

① $\boxed{\text{Limaçon}}$: $r=a\pm b\cos\theta$, $r=a\pm b\sin\theta$

$\boxed{\text{cardioid}}$: when $a=b$

Limaçons	cardioid: $a>b$, $a=b$, $a<b$
$(r,-\theta)$ x-axis symm: $r=a+b\cos\theta$	
$r=a-b\cos\theta$	
$(-r,-\theta)$ difficult, $(r,\pi-\theta)$, $\sin(\pi-\theta)=\sin(\theta-\pi)=\sin\theta$, symm y-axis: $r=a+b\sin\theta$	
$r=a-b\sin\theta$	



② $\boxed{\text{Lemniscate}}$ (Figure 8) $r^2=\pm a\cos 2\theta$, $r^2=\pm a\sin 2\theta$

$(r,-\theta)$ symm x-axis, $(r,\pi-\theta)$ symm y-axis, $(-r,\theta)$ symm origin

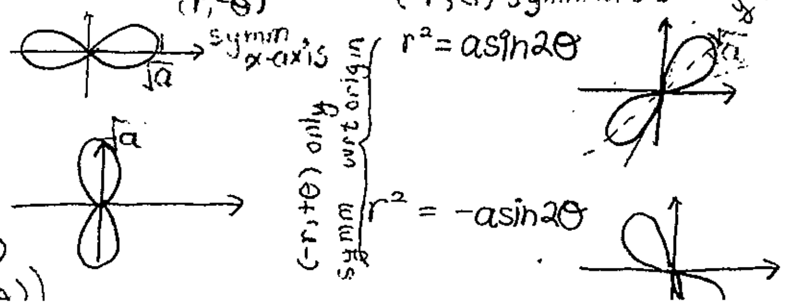
$r^2=a\cos 2\theta$ — $(r,-\theta)$ symm x-axis

$r^2=-a\cos 2\theta$

$(-r,\theta)$ symm wrt O

$r^2=a\sin 2\theta$

$(-r,+\theta)$ only symm wrt origin: $r^2=-a\sin 2\theta$



Sketch

$\boxed{ex}$ $y = 2\sin(3\theta)$

$\boxed{ex}$ $y = 3\sin(2\theta)$

$\boxed{ex}$ $r = 2 + 6\cos\theta$

$\boxed{ex}$ $r = 2 + 2\sin\theta$

$\boxed{ex}$ $r^2 = 4\cos 2\theta$

$\boxed{ex}$ $r^2 = -9\sin 2\theta$

Arithmetic in Polar Form (much easier for $\times$ $\div$)

Add/Subtr: $\mathbb{R}+\mathrm{Im}$.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$$

$$\boxed{z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]}$$

$$r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{r_1}{r_2}\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\Rightarrow \begin{cases} |z_1 z_2| = |z_1||z_2| \\ \arg(z_1 z_2) = \arg z_1 + \arg z_2 \end{cases} \qquad \begin{cases} \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \\ \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \end{cases}$$

$\therefore |z_1| = r_1 = \sqrt{r_1^2\underbrace{(\cos^2\theta_1+\sin^2\theta_1)}_{1}} = r_1$

$|z_2| = r_2$

etc.

$\bar z = x - iy$? $\arg(\bar z) \overset{??}{=} -\arg(z)$?

$\arg\left(\frac{1}{z}\right) = -\arg z$

Proof (w/o phasors)

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$
$$= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

HW
$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$
$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i(\sin(\theta_1-\theta_2))\right]$$

ex6

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt2 + i\sqrt2 \qquad 2e^{i\frac{\pi}{4}}$$

$$z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 5/2 + i\,5\sqrt3/2 \qquad 5e^{i\frac{\pi}{3}}$$

$$\to 10e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} \qquad \frac{2}{5}e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}$$

Given rectangular
a) mult & $\div$
b) To polar
c) $\times$ $\div$
Check

$$z_1 z_2 = 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \approx -2.588 + 9.659i$$

$$\frac{z_1}{z_2} = \frac{2}{5}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) \approx 0.3864 - 0.1035i$$

ex
$z_1 = -2+2i \quad\to |z_1| = 2\sqrt2 \quad \arg z_1 = \frac{3\pi}{4}$

$z_2 = 3i \quad |z_2| = 3 \quad \arg z_2 = \frac{\pi}{2}$

$$\Rightarrow |z_1 z_2| = |z_1||z_2| = \sqrt8\cdot\sqrt9 = 3\cdot 2\sqrt2 = 6\sqrt2$$

$$\left|\frac{z_1}{z_2}\right| = \frac{\sqrt8}{3} = \frac{2\sqrt2}{3}$$

Check: $z_1 z_2 = -6-6i \quad |z_1 z_2| = 6\sqrt{1^2+1^2} = 6\sqrt2 \quad \mathrm{Arg} = \frac{5\pi}{4}$

$$\frac{z_1}{z_2} = \frac{-2+2i}{3i}\cdot\frac{-3i}{-3i} = \frac{6+6i}{9} = \frac{2}{3} + \frac{2}{3}i \qquad \left|\frac{z_1}{z_2}\right| = \frac{2}{3}\sqrt2, \ \mathrm{Arg} = \frac{\pi}{4}$$

BONUS: Proof

$$\frac{1}{z} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$\mathrm{Im} = -\frac{y}{x^2+y^2}$

$\mathrm{Re} = \frac{x}{x^2+y^2}$

$$\mathrm{Arg}\left(\frac{1}{z}\right) = \tan^{-1}\left(\frac{-y}{x}\right) = -\tan^{-1}\left(\frac{y}{x}\right) = -\arg z$$

$$\frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

$$\mathrm{Arg}\left(\frac{1}{z}\right) = -\theta = -\mathrm{Arg}\,z$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$$

More Practice

8.3 (P.603)

7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 22,

23, 24, 29, 30, 32, 34, 37, 39, 48,

Hint $x+y=2$

50, 52, 53, 59, 63,

66, 67, 75, 76, 78, 83, 86, 89, 90,

* 94, 95, 96

See 90: $\theta_1 - \theta_2 = \alpha - (360^\circ + \underbrace{\beta}_{<0}) = \alpha - \underbrace{\beta}_{<0} = 90^\circ$ or

Kreyzig 12.2

7. $\left(\frac{6+8i}{4-3i}\right)^2 = 4(\cos\pi + i\sin\pi)$ Polar

Plot (6,8) 10, α 53°, 36.9°, (4,-3), 5 $\left(\frac{10}{5}e^{i\frac{\pi}{2}}\right)^2$

9. $\frac{2+i}{5-3i} = \sqrt{\frac{5}{34}}(\cos 1.00407 + i\sin 1.00407)$

$\frac{2+i}{5-3i}\cdot\frac{5+3i}{5+3i} = \frac{10-3+(5+6)i}{25+9} = \frac{7+11i}{34}$

$\frac{1}{34}\sqrt{49+121} = \frac{\sqrt{170}}{34}$

12.1 c) $\begin{cases} z_1 = 4+3i \\ z_2 = 2-5i \end{cases}$

~~3~~ $\mathrm{Re}(z_1^3) =$

$(\mathrm{Re}(z_1))^3 = \neq$

11 d) $\left\{ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} = \frac{-7-26i}{29} \right.$

a) $z_1 = 5e^{i\tan^{-1}\left(\frac{3}{4}\right)}$

$z_2 = \sqrt{29}\,e^{i\tan^{-1}\left(-\frac{5}{2}\right)}$

3, 4 $|z_1| = 5$

$\sqrt{4+25}$ $|z_2| = \sqrt{29}$

$\arg z_1 = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435$ rad

$\arg z_2 = \tan^{-1}\left(-\frac{5}{2}\right) = -1.1903$ rad

a) $\begin{cases} |z_1 z_2| = 5\sqrt{29} \\ \left|\frac{z_1}{z_2}\right| = \frac{5}{\sqrt{29}} \end{cases}$

b) $\begin{cases} \arg(z_1 z_2) = -0.5468 \\ \arg(z_1/z_2) = 1.8338 \end{cases}$

DeMoivre's Thm

$z = r(\cos\theta + i\sin\theta)$

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$$

ex7 $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos\frac{\pi}{4}\cdot 10 + i\sin\frac{10\pi}{4}\right)$

$r = \frac{1}{2}(\sqrt{2}) = \frac{1}{\sqrt{2}}$, $\frac{\pi}{4}$

$= \frac{1}{32}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$

$= \frac{1}{32}(0 + i\,1)$

$= \frac{1}{32}i$

$\frac{1}{2}+\frac{1}{2}i =$

$\begin{cases} \frac{1}{2}\sqrt{1^2+1^2} = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}} \\ \theta = \frac{\pi}{4} \end{cases}$

n^{th} Roots of Complex Numbers

$z = r(\cos\theta + i\sin\theta)$ $\quad \theta + 2k\pi$

$\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

$\frac{\theta}{n}$ $\quad \frac{\theta}{n}$

Plot

$\boxed{k = 0, 1, 2, \dots n-1}$ n distinct ω all different, more repeats

$r^{\frac{1}{n}}$, $r^{\frac{1}{n}}\sin\omega_k$, $r^{\frac{1}{n}}\cos\omega_k$

OR Polar

P.8 a)

$z_1 = 1+i = \sqrt{2}\,e^{i\pi/4}$

$z_2 = -1+i = \sqrt{2}\,e^{i3\pi/4}$

$z_1 z_2 = 2e^{i\pi} = 2$

$\frac{z_1}{z_2} = 1e^{i(-\pi/2)} = -i$

$z_1^{10} = 32e^{i\,5\pi/2}$

$z_1^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{i\frac{\pi}{16}}$ ← is only one answer

11.25° Smartboard $z_1 = \sqrt{2}e^{i(\pi/4+2k\pi)}$ 'invisible' overlap

→ $z_1^{\frac{1}{4}}$ circle into 4 equal parts, not overlap

$\boxed{\text{ex8}}$ $z = -64 = 64e^{i\pi}$

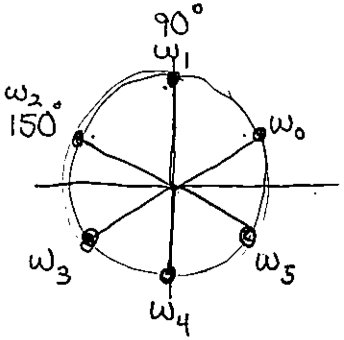
Find the 6^{th} roots

$$\sqrt[6]{z} = \underbrace{64^{\frac{1}{6}}}_{2}\left(\cos\frac{\pi + k2\pi}{6} + i\sin\frac{\pi+k2\pi}{6}\right) = 2e^{i\theta_k}$$

$k = 0, 1, 2, 3, 4, 5$

$\frac{\pi}{6} + k\frac{\pi}{3}$

$30^\circ + k\,60^\circ$



$\omega_0 = 2(\cos 30^\circ + i\sin 30^\circ)$

$= 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = \sqrt{3} + i$

$\omega_1 = 2(0 + i) = 2i$

$\omega_2 = -\sqrt{3} + i$

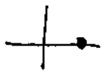
$\omega_3 = -\sqrt{3} - i$

$\omega_4 = -2i$

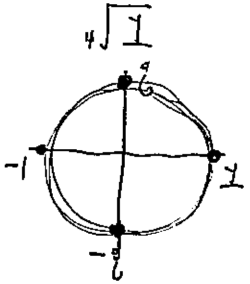
$\omega_5 = \sqrt{3} - i$

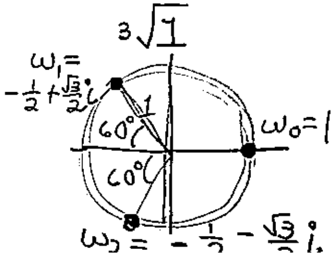
n^{th} roots of unity

$$\boxed{\sqrt[n]{1} = 1\left(\cos\frac{0 + 2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)}$$



$1 = 1e^{i0}$





$(-i)^4 = 1\,i^4 = 1\cdot 1 = 1$

$i^4 = i^2\,i^2 = 1$

Note: Cycle of i

i

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

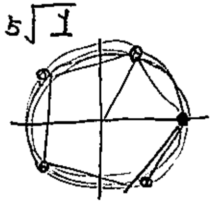
$i^5 = i$

$i^{27} = i^{24+3} = i^3 = -i$

$i^{108} = i^{4(27)} = 1$

$i^{110} = i^{108+2} = i^2 = -1$

$\frac{110}{4} = 27$, rem 2



$\frac{360^\circ}{5} = 72$

$\boxed{\text{ex6}}$ $z = 2 + 2i = 2\sqrt{2}e^{i\frac{\pi}{4}}$

$\sqrt[3]{z} = 8^{\frac{1}{2}\cdot\frac{1}{3}}\, e^{i\frac{\frac{\pi}{4}+2k\pi}{3}}$

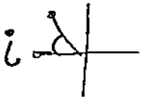
$= \sqrt{2}\, e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}$

$15^\circ + 120^\circ k \quad k = 0, 1, 2,$

$\omega_0 = \sqrt{2}(\cos 15^\circ + i\sin 15^\circ) \approx 1.366 + 0.366i$

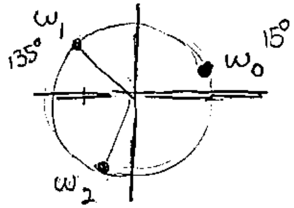
$\times 3 = 405 = 360 + 45$

$\omega_1 = \sqrt{2}(\cos \underset{-\frac{1}{\sqrt{2}}}{135^\circ} + i\sin \underset{\frac{1}{\sqrt{2}}}{135^\circ}) = -1 + i$



$\omega_2 = \sqrt{2}(\cos 255^\circ + i\sin 255^\circ) \approx -0.366 - 1.366i$

$\times 3 = 765 = 720 + 45$



$\boxed{\text{ex10}}$ Solve $z^6 + 64 = 0$

$z^6 = -64$

$z = \sqrt[6]{-64} = \sqrt[6]{2^6(-1)} = 2\sqrt[6]{-1}$

see ex8

$\boxed{\text{TI-89}}$

real mode $(-1)^{\frac{1}{6}}$ not real , $(3+2i)(4-2i) = 16+2i$

rectangular mode $(-1)^{\frac{1}{6}} = 0.866025 + 0.5i$

Polar mode $2+2i \rightarrow \sqrt{2}\, e^{i\frac{\pi}{4}}$

cSolve $(x\text{^}4 = -1 , x) \Rightarrow$ real mode exact

$$\begin{cases} \frac{\sqrt{2}}{2}(1+i) \\ \text{or } \frac{\sqrt{2}}{2}(1-i) \\ \frac{\sqrt{2}}{2}(-1+i) \\ \frac{\sqrt{2}}{2}(-1-i) \end{cases}$$

cSolve $(x\text{^}4 = 1 , x) \Rightarrow i , -i , 1 , -1$ ✓ ?

Kreyzig 12.6

complex exponential function $z = x + iy$

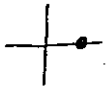
$$e^{z} \triangleq e^{x}(\cos y + i \sin y)$$

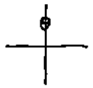
Properties $(e^{z})' = e^{z}$

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \text{ by trig ID}$$

$$\boxed{z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

standard Polar exponential polar form

$e^{2\pi i} = 1$  $e^{-i\pi/2} = -i$

$e^{\pi i/2} = i$  $e^{-i\pi} = -1$

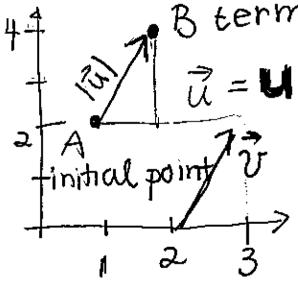
$e^{i\pi} = -1$ $\arg(e^{iy}) = y \pm 2k\pi$

$|e^{iy}| = 1 \because |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$

Go to Parametric first

(8.4) Vectors | Scalars

Vectors	Scalars
6th 9.1 directed qty	no direction, mass, time
displacement	distance
velocity	speed
acceleration	
force	
magnitude + direction	

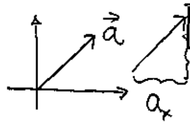


B terminal point

$\vec{u} = \mathbf{u} = \overrightarrow{AB} = \langle 1, 2 \rangle$

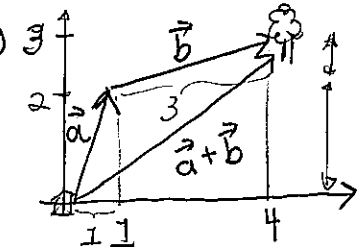
$\vec{v} = \vec{u}$ if direction & magnitude same

(length)
norm
absolute value



$\boxed{\vec{a} = \langle a_x, a_y \rangle = a_x\hat{i} + a_y\hat{j}}$

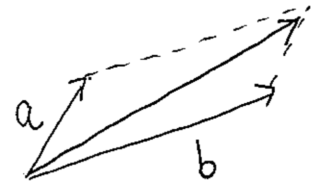
$\boxed{\text{Vector Addition}}$

① 

② Notice.

$$\vec{a} + \vec{b} \triangleq \langle a_x, a_y \rangle + \langle b_x, b_y \rangle$$
$$= \langle a_x + b_x , a_y + b_y \rangle$$

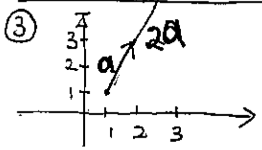
fits geometric interpretation

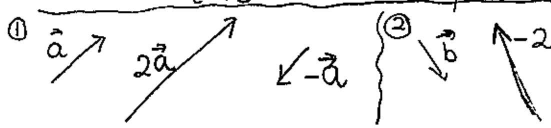
③ 

Parallelogram method

$\boxed{\text{Scalar Multiplication}}$ $c\vec{a} = \langle ca_x , ca_y \rangle$

c<0 means flip

③ 



10.7 Plane Curves & Parametric Eqn

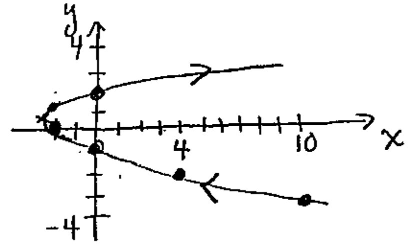
(maybe not function)

$(f(t), g(t))$ BUG flying. t = time

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ Parameter $t \in I$

ex1 $\begin{cases} x(t) = t^2 - 3t \\ y(t) = t - 1 \end{cases}$

t	x	y
-2	10	-3
-1	4	-2
0	0	-1
1	-2	0
2	-2	1
3	0	2



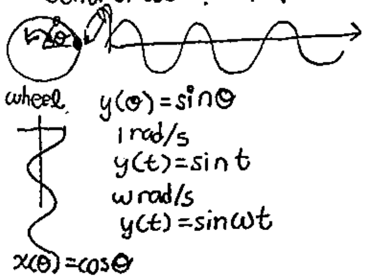
$t \to 2t$ $\boxed{\text{same thing TWICE FAST}}$ ★

$\begin{cases} x(t) = 4t^2 - 6t \\ y(t) = 2t - 1 \end{cases}$

see pg 802

Parametrization < curve, direction, speed

• Defn. of $\cos\theta$? $\sin\theta$?



$y(\theta) = \sin\theta$
1 rad/s
$y(t) = \sin t$
ω rad/s
$y(t) = \sin \omega t$

HOW TO SKETCH

① eliminate the parameter
- plug
- $\sin^2\theta + \cos^2\theta = 1$

② Check the restrictions/direction

ex2 $\begin{cases} x(t) = t^2 - 3t \\ y(t) = t - 1 \end{cases}$ $\rightarrow$ $x = (y+1)^2 - 3(y+1)$

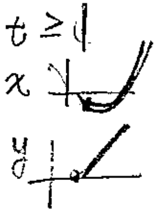
$= y^2 + 2y + 1 - 3y - 3$

$x = y^2 - y - 2$

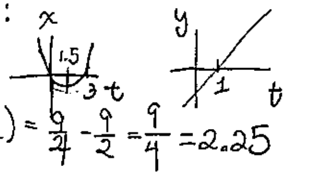
$x = (y-2)(y+1)$

• direction: $t \uparrow \Rightarrow x\uparrow, y\uparrow$

$t \geq 1$



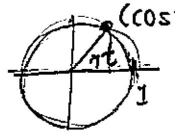
• Restriction:



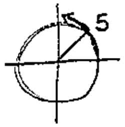
$y \in \mathbb{R}$

$x(\tfrac{3}{2}) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4} = -2.25$ $\quad x \geq -2.25$

ex3 $\left[\begin{array}{l} x = \cos t \quad y = \sin t \\ t \in [0, 2\pi] \end{array} \right.$ ← Just the UNIT CIRCLE!



$\begin{cases} x = 5\cos t \\ y = 5\sin t \end{cases}$



t = seconds
1 rad/second

$\begin{cases} x = 5\cos(3t) \\ y = 5\sin(3t) \end{cases}$ ← • speed angular 3 rad/second?

• speed 3cm/s

$\begin{cases} x = 5\cos(0.6t) \\ y = 5\sin(0.6t) \end{cases}$

$s = r\theta$

$\frac{ds}{dt} = r\frac{d\theta}{dt}$

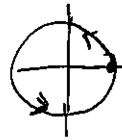
$\leftarrow v = r\omega$

$\omega = \frac{3}{5} = 0.6$ rad/s

eliminate param:

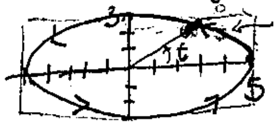
$\sin^2 t + \cos^2 t = 1$

$y^2 + x^2 = 1$ CIRCLE



ex $x = 5\cos t$ $\quad y = 3\sin t$ $\Rightarrow$ ellipse! OBLONG CIRCLE

$t \in [0, 2\pi)$

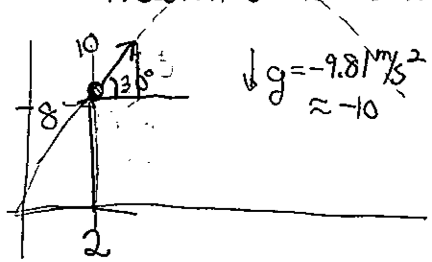


$\cos^2 t + \sin^2 t = 1$ $\quad -5 \leq x \leq 5$ ✓

$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ $\quad -3 \leq y \leq 3$ ✓

ex PROJECTILE (extension or spongebob class activity)

Newton's 1st Law: inertia



$\downarrow g = -9.81\,m/s^2 \approx -10$

$\begin{cases} x(t) = 5\sqrt{3}\,t + 2 \\ y(t) = 8 + 5t - \frac{9.81}{2}t^2 \end{cases}$

Is it a parabola?

$\begin{cases} x(t) = v_0 \boxed{t} + x_0 \\ y(t) = \frac{1}{2}gt^2 + v_y t + y_0 \end{cases}$

YES!

$y(t) = \frac{1}{2}gt^2 + v_0 t + y_0$

STOP MOTION camera?

ex4 Sketch

$$\begin{cases} x(t) = \sin t \\ y(t) = 2 - \cos t \end{cases}$$

can see here it's a circle

$\sin^2 t + \cos^2 t = 1$

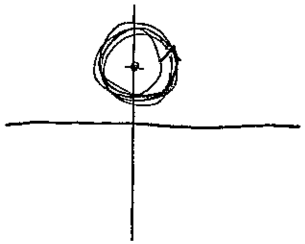
$x^2 + (2-y)^2 = 1$ ← ellipse / hyp / parab

$x^2 + 4 - 4y + y^2 = 1$

$x^2 + (y^2 - 4y + 4) = 1 - 4 + 4$

$x^2 + (y-2)^2 = 1$

circle



ex4

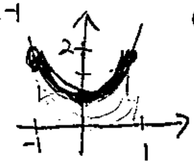
$$\begin{cases} x(t) = \sin t \\ y(t) = 2 - \cos^2 t \end{cases}$$

$0 \le \cos^2 t \le 1$

$2 - 0 \ge 2 - \cos^2 t \ge 2 - 1$

① $x^2 + (2-y) = 1$

$y = x^2 + 1$



② Restrictions!

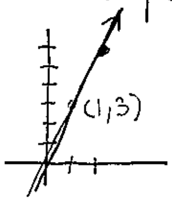
$-1 \le x \le 1$

$1 \le y \le 2$

$t=0: \ x = 0, \ y = 1$

ex5 Parametric Eqn of a Line

Slope 3 thru (2,6)



$y = 3(x-2) + 6$

$$\begin{cases} x = t \\ y = 3(t-2) + 6 \end{cases}$$

$$\begin{cases} x = \boxed{t+2} \\ y = \boxed{3}(t) \boxed{+6} \end{cases}$$

$$\begin{cases} x = 5t \\ y = 15t \end{cases}$$

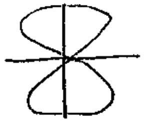
$x = 1 + 2t$

$y = 3 + 6t$?

ex7 Graphing CALC

$-1 \le x = \sin 2t \le 1$

$-2 \le y = 2\cos t \le 2$



ex8 Polar as Param.

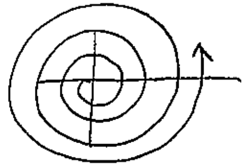
$r = \theta$, $1 \le \theta \le 10\pi$

Graph CALC

$x = \theta \cos\theta$

$y = \theta \sin\theta$

$[-32, 32]$

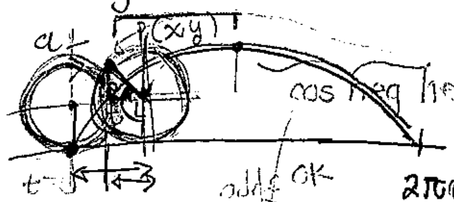


See Applet

AP Calc BC

AP Phys C

ex6 Cycloid



$t \in [0, 2\pi)$

cos is positive

cos neg here

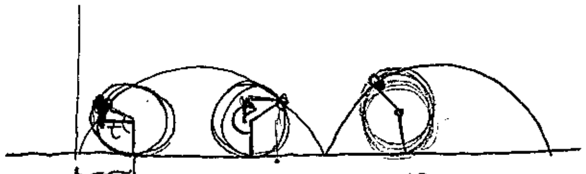
add OK

$2\pi a$

$$\begin{cases} x = at + a\cos(t - 90^\circ) = at - a\sin t = a(t - \sin t) \\ y = a + a\sin(t - 90^\circ) = a + a\cos t = a(1 - \cos t) \end{cases}$$

$-\underbrace{\sin(90 - t)}_{\cos t}$

Show rotate on smartboard. Unit Circle



axis x

Look from other side of paper

①

$x = at - a\underbrace{\sin t}_{\text{positive}}$ $s = r\theta$

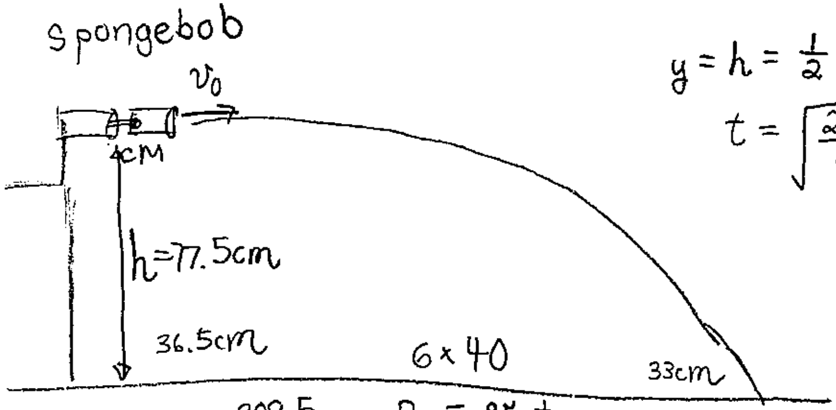
$y = a - a\underbrace{\cos t}_{\text{negative}}$

$x(t) = \underbrace{at}_{\text{add}} - a\underbrace{\sin t}_{\text{neg}}$ $s = r\theta$

$y(t) = a - a\underbrace{\cos t}_{\text{adds (neg)}}$

Same as ① at ← $2\pi + \frac{3\pi}{4}$

$s = r\theta$ ok



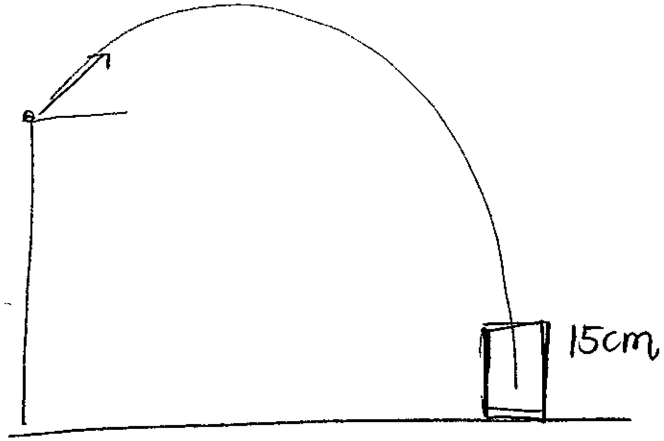
$$y = h = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2h}{g}}$$

$$309.5 = R = v_0 t$$

$$v_0 = \frac{R}{t} = R\sqrt{\frac{g}{2h}}$$

$$= 778.23 \text{ cm/s}$$



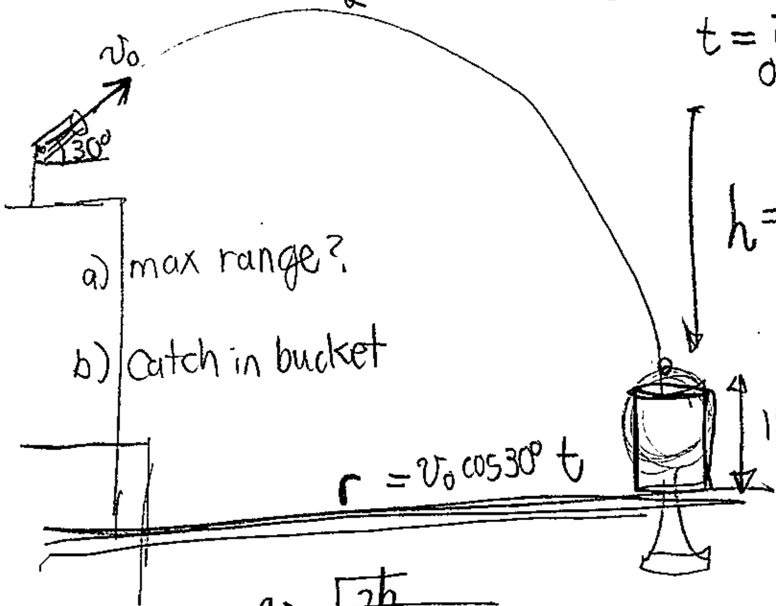
$$y(t) = \underset{77.5}{y_0} + v_{0y} t - \frac{1}{2} g t^2 = 15$$

$$t = -0.136989$$
$$0.931102$$

$$x(t) = \cancel{778.}\ v_0 \cos 30^\circ t = \boxed{627.532 \text{cm}}$$

$$\cancel{y_0} + v_0 \sin 30^\circ t - \frac{1}{2} 980 t^2 = 62.5 \text{cm}$$

$$t = \cancel{0.040628}$$
$$0.753485$$



a) max range?

b) Catch in bucket

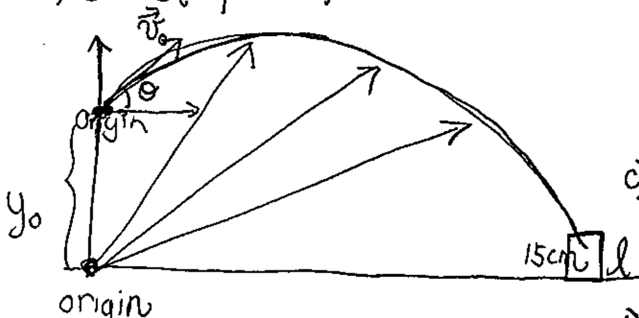
$$r = v_0 \cos 30^\circ t$$

$$\cancel{r = v_0 \sqrt{\frac{2h}{g}}} =$$

Measure θ, y_0, bucket l

$v_0 = 778.23$ cm/s. Launch at θ

a) sketch path, vectors



b) Parametrically
$$\begin{cases} x(t) \\ y(t) \end{cases} \quad \begin{matrix} \vec{r}(0.3) \\ \vec{v}(0.3) \end{matrix}$$
★ Speed

c) Range
max height

d) Where put bucket

e) Verify path is parabola

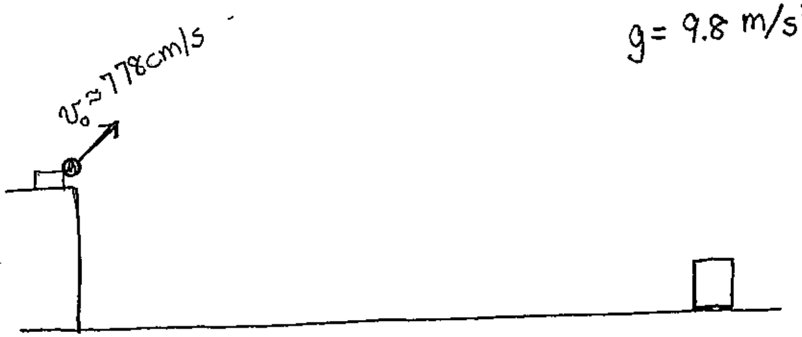
$$\vec{v}_0 = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$$

$$\vec{r}(t) = \langle v_0 \cos\theta\ t + x_0,\ y_0 + v_0 \sin t - \tfrac{1}{2} 9.8 t^2 \rangle$$

Ch 9 Projectile Worksheet

NAME:

$g = 9.8 \text{ m/s}^2$



• where should you place the bucket so that it catches the projectile?

a) Take the necessary measurements. Label them (including units) on the diagram above.

b) Sketch the projectile's trajectory on the diagram.

c) Sketch the horizontal and vertical components of $\vec{v}_0$.

d) $\vec{v}_0 =$ ____________________

e) Parametrically express the trajectory.

$x(t) =$

$y(t) =$

f) Express the trajectory using a position vector.

$\vec{r}(t) =$

h) Predict where the bucket should be placed.

i) Predict the maximum height of the projectile.

j) Verify the projectile's path is a parabola.

k) Bonus: (+15 HW). Carefully take measurements for $\vec{v}_0$ so that your parametric equations actually allow the bucket to catch the projectile.

Ch 10 rref classwork

- Your system of Equations:

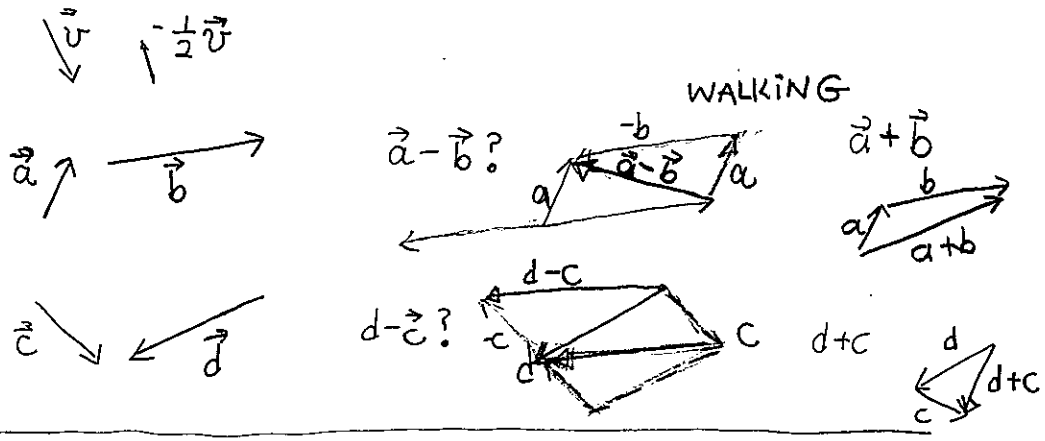
$$\begin{cases} \\ \\ \end{cases}$$

- Use Gauss-Jordan elimination to get your system of equations in <u>reduced</u> row echelon form.
  Show "$\underrightarrow{R_1+R_2}$" for each step.

- what is your solution? ______________________

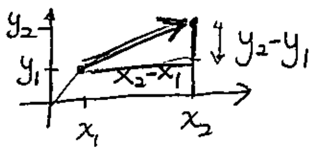
$\boxed{\text{Vector Subtraction}}$ $\vec{a}-\vec{b} = \langle a_x - b_x, a_y - b_y \rangle$

(6th 9.1)

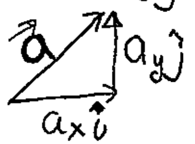


- $\vec{v}$ $P(x_1, y_1) \longrightarrow Q(x_2, y_2)$

  $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$



- $\vec{a} = a_x \hat{i} + a_y \hat{j}$ Addition makes sense



$\boxed{\text{ex1}}$ $P(-2,5)$ initial pt $\quad Q(3,7)$ terminal Pt

a) vector $\vec{u} = \underline{\langle 5, 2 \rangle} = 5\hat{i} + 2\hat{j}$

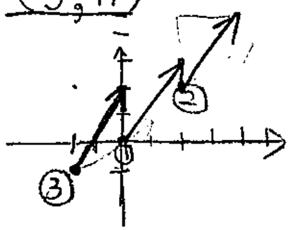
b) $\vec{V} = \langle 3, 7 \rangle$

initial $(2,4)$ terminal at $\underline{(5, 11)}$

c) $\vec{w} = \langle 2, 3 \rangle$

Sketch with initial pts

$(0,0)$, $(2,2)$, $(-2,-1)$, $(1,4)$
① ② ③



$$\left\{ \begin{aligned} &\vec{v} = \langle v_1, v_2 \rangle = v_1\hat{i} + v_2\hat{j} \\ &|\vec{v}| = \sqrt{v_1^2 + v_2^2} \;(= \vec{v}\cdot\vec{v}) \\ &\vec{u} \pm \vec{v} = \langle u_1 \pm v_1, u_2 \pm v_2 \rangle \\ &c\vec{u} = \langle cu_1, cu_2 \rangle \end{aligned} \right.$$

See properties on Pg 611 (6th 583)

unit vector $\left\{ \begin{aligned} \hat{i} &= \langle 1, 0, 0 \rangle \\ \hat{j} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned} \right.$

$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

$\boxed{\text{ex3}}$ $\vec{u} = \langle 2, -3 \rangle$

$\vec{v} = \langle -1, 2 \rangle$

$\vec{u} + \vec{v} = \langle 1, -1 \rangle \quad 2\vec{u} - 3\vec{v} = \langle 4, -6 \rangle - \langle -3, 6 \rangle = \langle 7, -12 \rangle$

$\vec{u} - \vec{v} = \langle 3, -5 \rangle \quad 2\vec{u} + 3\vec{v} = \langle 1, 0 \rangle$

$\boxed{\text{ex4}}$ a) $\vec{u} = \langle 5, -8 \rangle = 5\hat{i} - 8\hat{j}$

b) $\vec{u} = 3\hat{i} + 2\hat{j}$

$\vec{v} = -\hat{i} + 6\hat{j}$

$2\vec{u} + 5\vec{v} = 2(3\hat{i} + 2\hat{j}) + 5(-\hat{i} + 6\hat{j})$

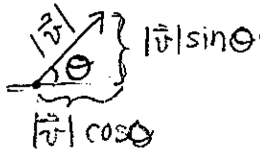
$= 6\hat{i} + 4\hat{j} - 5\hat{i} + 30\hat{j}$

$= \hat{i} + 34\hat{j}$

Horizontal & Vertical Components "DIRECTION"

when $|\vec{v}|$ & θ are known (θ = from +x axis)

(force, geometrically)



like polar

$\Rightarrow \vec{v} = \langle |\vec{v}|\cos\theta, |\vec{v}|\sin\theta \rangle$

$= \underbrace{|\vec{v}|\cos\theta}\,\hat{i} + |\vec{v}|\sin\theta\,\hat{j}$

"Projection" onto x-axis

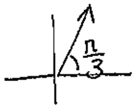
Horizontal component

$\boxed{\text{ex5}}$ a) $\vec{v} = 4\langle 1, \sqrt{3}\rangle = 4\hat{i} + 4\sqrt{3}\hat{j}$

Length $= 8$

Direction $= \frac{\pi}{3}$

Horiz Vertical Components?



$v_x = |\vec{v}|\cos\frac{\pi}{3} = 8 \times \frac{1}{2}$

$v_y = |\vec{v}|\sin\frac{\pi}{3} = 8\ \frac{\sqrt{3}}{2}$

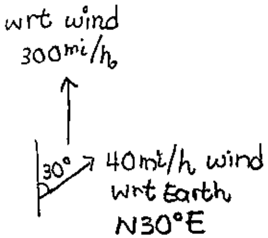
b) $\vec{u} = -\sqrt{3}\,\hat{i} + \hat{j}$

Direction?

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = \frac{\pi}{6}$

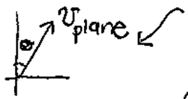
$\theta = \boxed{\frac{5\pi}{6}}$

$\boxed{\text{ex6}}$



a) $\vec{v}_{\substack{\text{plane wrt}\\ \text{air}}} = \langle 0, 300\rangle$

$\vec{u}_{air} = \langle 20, 20\sqrt{3}\rangle$



b) $\vec{v}_{plane} = \vec{v}_{p\,wrt\,air} + \vec{u}_{air}$

$= \langle 20, 300 + 20\sqrt{3}\rangle$

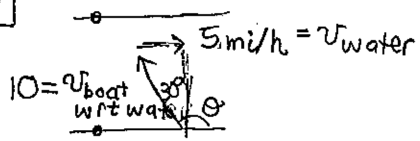
c) $|\vec{v}_{plane}| = 335.2$ mi/h

$\theta = \tan^{-1}\left(\frac{20}{300+20\sqrt{3}}\right)$

$= 3.4^\circ$

$\boxed{\text{N } 3.4^\circ \text{ E}}$

$\boxed{\text{ex7}}$



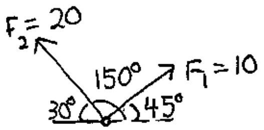
$v_{boat}\cos\theta = -5$

$\cos\theta = -\frac{1}{2}$

$\theta = 120^\circ$

$\boxed{\text{N } 30^\circ \text{ W}}$

$\boxed{\text{ex8}}$ Resultant Force



$\vec{F} = \langle 10\cos 45^\circ - 20\cos 30^\circ, 5\sqrt{2} + 10\rangle$

$= \langle 5\sqrt{2} - 10\sqrt{3}, 5\sqrt{2} + 10\rangle$

8.5 Dot Product

6th 9.2

$\vec{u} = \langle u_1, u_2 \rangle$

$\vec{v} = v_1 \hat{i} + v_2 \hat{j}$

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 \quad \text{scalar}} = |\vec{u}||\vec{v}| \cos\theta$$

↑ ↑ vectors

$\theta \in [0, 180°]$

smaller angle between

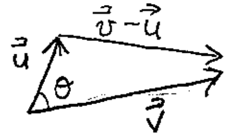
NOT α

DOT Product Thm

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta}$$

Find the angle θ? solve $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

Proof



Law of Cosines

$|\vec{v} - \vec{u}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos\theta$

$(\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u})$

$= (\vec{v} - \vec{u}) \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot \vec{u}$ Prop 3

$= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{u}$

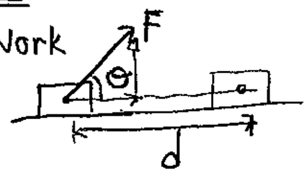
$\therefore = \cancel{|\vec{v}|^2} - \cancel{2}\vec{u} \cdot \vec{v} + \cancel{|\vec{u}|^2}$

$= \cancel{|\vec{v}|^2} + \cancel{|\vec{u}|^2} - \cancel{2}|\vec{u}||\vec{v}| \cos\theta$

$\therefore \vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

- Uses

① Work

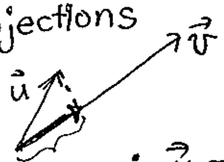


$W = \vec{F} \cdot \vec{d}$

$= (F\cos\theta) d$ Nm

Joules

② Projections



$$\text{proj}_{\vec{v}} \vec{u} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}}_{|\vec{u}|\cos\theta} \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{unit vector}}$$

- Project $\vec{u}$ onto $\vec{v}$ & $\perp$ $\vec{v}$

(every vector can be decomposed to orthogonal components)

ex2 $\vec{u} = \langle 2, 5 \rangle$

$\vec{v} = \langle 4, -3 \rangle$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{8 - 15}{\sqrt{29 \cdot 25}} = \dfrac{-7}{5\sqrt{29}}$

$\theta = \cos^{-1}\left(\dfrac{-7}{5\sqrt{29}}\right) \approx \boxed{105.1°}$

corrollaries

① ORTHOGONAL (NORMAL) PERPENDICULAR

$$\boxed{\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0}$$

proof "$\Rightarrow$" $\theta = 90° \Rightarrow \vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos 90° = 0$

"$\Leftarrow$" $|\vec{u}||\vec{v}| \cos\theta = 0 \Rightarrow \theta = 90°$

ex1

b) $\vec{u} = 2\hat{i} + \hat{j}$

$\vec{v} = 5\hat{i} - 6\hat{j}$

$\vec{u} \cdot \vec{v} = 2 \times 5 + 1(-6) = 4$ NOT a vector anymore!

ex3 a) $\vec{u} = \langle 3, 5 \rangle$ ORthogonal?

$\vec{v} = \langle 2, -8 \rangle$ $\quad \vec{u} \cdot \vec{v} = -34 \neq 0$ NO

b) $\langle 2, 1 \rangle \cdot \langle -1, 2 \rangle = 0$ yes

Properties

in HW prove

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot c\vec{v} = c(\vec{u} \cdot \vec{v})$

$\langle cu_1, cu_2 \rangle \cdot \langle v_1, v_2 \rangle$

$= cu_1 v_1 + c u_2 v_2$

$= u_1 c v_1 + u_2 c v_2 = \langle u_1, u_2 \rangle \cdot \langle cv_1, cv_2 \rangle$

$= c(u_1 v_1 + u_2 v_2) = c(\vec{u} \cdot \vec{v})$

$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

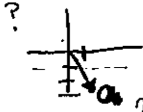
$|\vec{u}|^2 = \vec{u} \cdot \vec{u} \quad \leftarrow \quad |\vec{u}|^2 = u_1^2 + u_2^2 = \vec{u} \cdot \vec{u}$

★ ex ② Finding $\perp$ vector

$\vec{a} = \langle 1, -3 \rangle$

$\vec{b} = \langle 3, +1 \rangle \quad \vec{b} \perp \vec{a}$ switch x, y flip one sign

Why?



$m_b = \dfrac{+1 \text{ up}}{3 \text{ right}} \Rightarrow \vec{b} = \langle 3, 1 \rangle$

$m_a = -\dfrac{3}{1}$

$\boxed{\text{ex}}$ $\vec{a} = \langle 2, 3 \rangle$. Find $\perp$ vector

$\vec{b} = \underline{\langle -3, 2 \rangle}$ DOT = 0 too

Find $\perp$ UNIT vector

$$\vec{c} = \left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle = \frac{\vec{b}}{|\vec{b}|} \text{ normalized}$$

$\boxed{\text{ex}}$

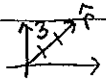
$\vec{a} = \langle -3, 4 \rangle$

find a <u>unit</u> vector $\perp \vec{a}$

$$\vec{b} = \frac{\langle -4, -3 \rangle}{5} = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle$$

---

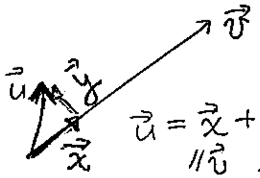
③



$\boxed{\text{component}}$ of $\vec{u}$ along $\vec{v}$

(scalar length)

$$\frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|} = |\vec{u}| \cos\theta$$



$\vec{u} = \vec{x} + \vec{y}$

$\parallel \vec{v}$ $\perp \vec{v}$

$\boxed{\text{PROJECTION of } \vec{u} \text{ onto } \vec{v}}$

VECTOR

$$\boxed{\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} = \vec{x} \\ \vec{y} &= \vec{u} - \vec{x} \end{aligned}}$$

$\boxed{\text{ex6}}$ $\vec{u} = \langle -2, 9 \rangle$

$\vec{v} = \langle -1, 2 \rangle$

ASK ① draw

② $\text{proj}_{\vec{v}} \vec{u}$ is what in pic? ($\vec{x}$)

③ component of $\vec{u} \perp \vec{v}$ is?

a) $\text{proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \dfrac{2+18}{5} \langle -1, 2 \rangle = \langle -4, 8 \rangle$

b) Resolve $\vec{u}$ into $\vec{x}$ parallel to $\vec{v}$

& $\vec{y}$ orthogonal to $\vec{v}$

$\vec{x} = \langle -4, 8 \rangle$

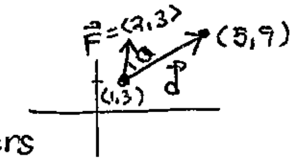
$\vec{y} = \vec{u} - \vec{x} = \langle -2, 9 \rangle - \langle -4, 8 \rangle$

$= \langle 2, 1 \rangle$

CW # 3, 12, 18, 20, 25

$\boxed{\text{Work}}$ done by force in moving along $\vec{D}$

$$W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos\theta$$



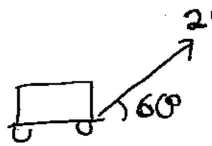
$\boxed{\text{ex7}}$ $\vec{F} = \langle 2, 3 \rangle$ Newtons

move from $(1, 3)$ to $(5, 9)$ meters

$$W = \vec{F} \cdot \vec{d} = \langle 2, 3 \rangle \cdot \langle 4, 6 \rangle = 8 + 18 = \boxed{26 \text{ J}}$$

N·m

$\boxed{\text{ex8}}$



move 100 m

$W = (20 \cos 60°) \text{N}\, 100 \text{m}$

$= 20 \times \frac{1}{2} \times 100$ Nm

$= 1000$ J

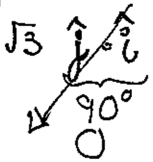
OR $W = \vec{F} \cdot \vec{d}$

$= \langle 20 \cos 60°, 20 \sin 60° \rangle \cdot \langle 100, 0 \rangle$

$= (10\hat{i} + 10\sqrt{3}\hat{j}) \cdot (100\hat{i})$

$= 10\hat{i} \cdot 100\hat{i} + 10\sqrt{3}\hat{j} \cdot 100\hat{i}$

$= 1000 \underbrace{\hat{i} \cdot \hat{i}}_{|1|^2} + 1000\sqrt{3}\, \hat{j} \cdot \hat{i}$



$= 1000$

DEMOS CW: 30, 36, 38, 46, 47

(36) $(\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v}) \overset{\text{Proof}}{=} |\vec{u}|^2-|\vec{v}|^2$

41

$(\vec{u}-\vec{v})\cdot\vec{u} + (\vec{u}-\vec{v})\cdot\vec{v}$

$= \vec{u}\cdot\vec{u} - \cancel{\vec{v}\cdot\vec{u}} + \cancel{\vec{u}\cdot\vec{v}} - \vec{v}\cdot\vec{v}$

(38) $\vec{v}\cdot \text{proj}_{\vec{v}}\vec{u} = \vec{v}\cdot\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}\right) = \frac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{v}|^2}}\,\cancel{\vec{v}\cdot\vec{v}} = \vec{u}\cdot\vec{v}$

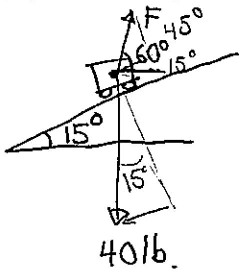
44

Guess: $\underbrace{|\text{proj}|}_{|\vec{u}|\cos\theta}|\vec{v}|\cos 0^\circ$

$= \vec{u}\cdot\vec{v}$

(46) Like HW 43

52



Keep from rolling down $|\vec{F}| = ?$

$|\vec{F}|\cos 45^\circ = 40\sin 15^\circ$ $\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$|\vec{F}|\cdot\frac{\sqrt{2}}{2} = 40 \oplus \sqrt{\frac{1-\cos 30^\circ}{2}}$

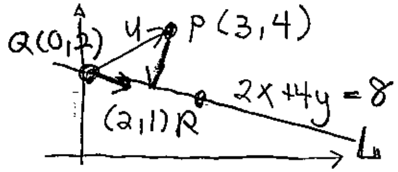
$|\vec{F}|\frac{1}{\sqrt{2}} = 40\sqrt{\frac{1-\sqrt{3}/2}{2}}$

$|\vec{F}| = 40\sqrt{\frac{1-\sqrt{3}/2}{\cancel{2}}} = 40\sqrt{1-\frac{\sqrt{3}}{2}}$

≈ 14.641 ?

53

(47) distance between Point & Line (Hint: projection ⊥)



$d(P, L) = \|\vec{u}_\perp\|$

$\vec{u}_\perp = \vec{u} - \vec{u}_{\parallel}$

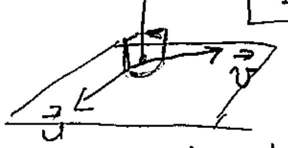
$\vec{u}_{\parallel} = \text{proj}_{\vec{v}}\vec{u}$

(6th ed Stewart 9.5)

Anton Calculus 11.4 Cross Product

$$\boxed{\underbrace{\vec{u}\times\vec{v}}_{\text{vector product}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

★ $\boxed{\vec{u}\times\vec{v} \perp \vec{u} \ \& \ \vec{v}}$



Right-hand Rule

$\boxed{\text{ex1}}$ $\vec{u} = \langle 1, 2, -2\rangle$

$\vec{v} = \langle 3, 0, 1\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \hat{i}(2+0) - \hat{j}(1+6) + \hat{k}(0-6)$$

$$= \langle 2, -7, -6\rangle$$

Properties

a) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ NOT COMMUTATIVE interchange 2 rows in determinant makes it negative!

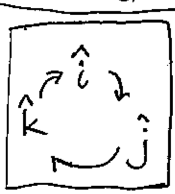
b) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$ distributive, Keep order

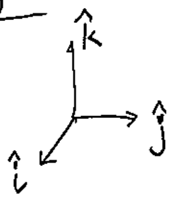
c) $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$

d) $k(\vec{u}\times\vec{v}) = (ku)\times\vec{v} = \vec{u}\times(k\vec{v})$

e) $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$

f) $\vec{u}\times\vec{u} = \vec{0}$





$\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

$\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$

WARNING: not associative!

$$\hat{i}\times(\hat{j}\times\hat{j}) = \vec{0} \neq \underbrace{(\hat{i}\times\hat{j})}_{\hat{k}}\times\hat{j} = -\hat{i}$$

(ex1) $(1\hat{i}+2\hat{j}-2\hat{k})\times(3\hat{i}+\hat{k})$

$= \cancel{3\hat{i}\times\hat{i}} + \hat{i}\times\hat{k} + 6\hat{j}\times\hat{i} + 3\hat{j}\times\hat{k} - 6\hat{k}\times\hat{i} - \cancel{2\hat{k}\times\hat{k}} = 3\hat{i} - 7\hat{j} - 6\hat{k}$

$\quad -\hat{j} + -6\hat{k} + 3\hat{i} - 6\hat{j}$

Thm

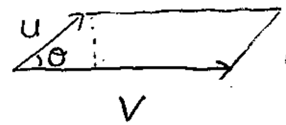
$$\begin{cases} \vec{u}\cdot\underbrace{(\vec{u}\times\vec{v})}_{\perp\vec{u}} = 0 \\ \vec{v}\cdot(\vec{u}\times\vec{v}) = 0 \end{cases}$$

$\boxed{\text{ex3}}$ find a vector orthogonal to both $\vec{u} = \langle 2, -1, 3\rangle$ and $\vec{v} = \langle -7, 2, -1\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix} = -5\hat{i} - 19\hat{j} - 3\hat{k}$$

Thm

a) $\boxed{\|\vec{u}\times\vec{v}\| = |\vec{u}||\vec{v}|\sin\theta}$

b) 

$\boxed{\text{Area} = |\vec{u}\times\vec{v}|}$

c) $\boxed{\vec{u}\times\vec{v} = 0 \iff \vec{u} \parallel \vec{v},\ \vec{u} = c\vec{v}}$ $\quad \sin\theta = 0 \Longrightarrow$

Proof (a) Bonus: Hint RHS → cos, proj → determinant

$$|\vec{u}||\vec{v}|\sin\theta = |u||V|\sqrt{1-\cos^2\theta}$$

$$= |u||v|\sqrt{1 - \frac{(\vec{u}\cdot\vec{v})^2}{|u|^2|v|^2}}$$

$$= \sqrt{|u|^2|v|^2 - (\vec{u}\cdot\vec{v})^2}$$

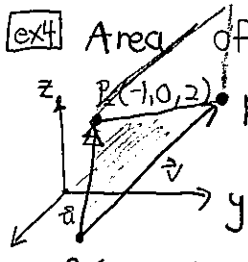
$$= \sqrt{(u_1^2+u_2^2+u_3^3)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2}$$

$$= \sqrt{(u_2v_3-u_3v_2)^2 + (u_1v_3-u_3v_1)^2 + (u_1v_2-u_2v_1)^2}$$

↓ matrix determinant

$$= |\vec{u}\times\vec{v}|$$

ex4 Area of triangle?

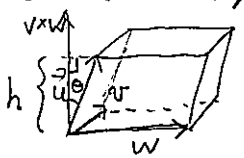


$$A = \frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & +2 \\ -2 & +2 & +3 \end{vmatrix}\right|$$

$$= \frac{15}{2}$$

Scalar Triple Product

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{u}\cdot\underbrace{\vec{v}\times\vec{w}}_{1^{st}\text{ of course}} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



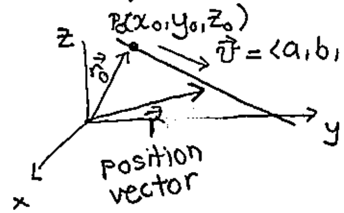
- instead of $\hat{i}$, component $\times u_1$ etc.

ex5 Calculate the scalar triple product $\vec{u}\cdot(\vec{v}\times\vec{w})$

Scan & 11.6 planes in 3space +HW

stewart

9.6 Equation of a line. Give point and parallel direction



$$\{\vec{r} : \vec{r}-\vec{r_0} = t\vec{v}\}$$

$$\boxed{\vec{r} = \vec{r_0} + \vec{v}t}$$
$$= \langle x_0, y_0, z_0\rangle + \langle a,b,c\rangle t$$

Parametric equations for a line

$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}$$

ex1 Find for line thru $(5,-2,3)$

$\vec{v} = \langle 3,-4,2\rangle$

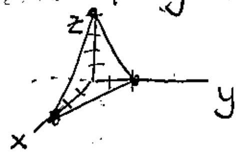
$$\vec{r} - \vec{r_0} = \vec{v}t$$

$$\vec{r} = \langle 5,-2,3\rangle + \langle 3,-4,2\rangle t$$

Ch8

Review: eqn of plane

$4x + 6y + 3z = 12$



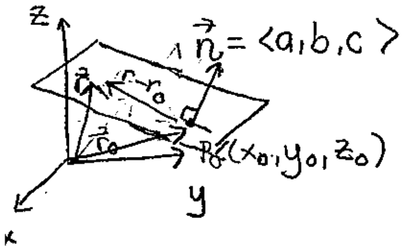
$y = z = 0 \Rightarrow x = 3$

$x = y = 0 \Rightarrow z = 4$

$x = z = 0 \Rightarrow y = +2$

Equation of Plane

$\begin{cases} \text{Given point} \\ \text{normal vector} \end{cases}$



$\{\vec{r} : (\vec{r} - \vec{r_0}) \perp \vec{n}\}$

$$\boxed{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}$$

$(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) \cdot \langle a, b, c \rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\boxed{\text{ex3}}$ Eqn of plane $\vec{n} = \langle 4, -6, 3 \rangle$

$P\ (3, -1, -2)$

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$

$4(x - 3) - 6(y + 1) + 3(z + 2) = 0$

$4x - 6y + 3z = 12$

Vector Worksheet

Reference: Precalculus 6th Edition by Stewart, Redlin, Watson

9.2 Find the component of u along v

27. $\mathbf{u} = 7\mathbf{i}$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$

29–34 ■ (a) Calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$. (b) Resolve $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

29. $\mathbf{u} = \langle -2, 4 \rangle$, $\mathbf{v} = \langle 1, 1 \rangle$

DISCOVERY ■ DISCUSSION ■ WRITING

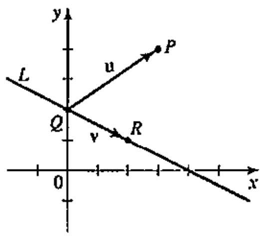
53. **Distance from a Point to a Line** Let L be the line $2x + 4y = 8$ and let P be the point $(3, 4)$.

(a) Show that the points $Q(0, 2)$ and $R(2, 1)$ lie on L.

(b) Let $\mathbf{u} = \overrightarrow{QP}$ and $\mathbf{v} = \overrightarrow{QR}$, as shown in the figure. Find $\mathbf{w} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$.

(c) Sketch a graph that explains why $|\mathbf{u} - \mathbf{w}|$ is the distance from P to L. Find this distance.

(d) Write a short paragraph describing the steps you would take to find the distance from a given point to a given line.



29–32 ■ Determine whether or not the given vectors are perpendicular.

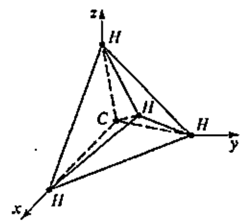
9.4 29. $\langle 4, -2, -4 \rangle$, $\langle 1, -2, 2 \rangle$ 30. $4\mathbf{j} - \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$

48. **Central Angle of a Tetrahedron** A *tetrahedron* is a solid with four triangular faces, four vertices, and six edges, as shown in the figure. In a *regular* tetrahedron, the edges are all of the same length. Consider the tetrahedron with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, and $D(1, 1, 1)$.

(a) Show that the tetrahedron is regular.

(b) The center of the tetrahedron is the point $E(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (the "average" of the vertices). Find the angle between the vectors that join the center to any two of the vertices (for instance, $\angle AEB$). This angle is called the *central angle* of the tetrahedron.

NOTE: In a molecule of methane (CH_4) the four hydrogen atoms form the vertices of a regular tetrahedron with the carbon atom at the center. In this case chemists refer to the central angle as the *bond angle*. In the figure, the tetrahedron in the exercise is shown, with the vertices labeled *H* for hydrogen, and the center labeled *C* for carbon.



9.5

17–20 ■ Find a vector that is perpendicular to the plane passing through the three given points.

17. $P(0, 1, 0)$, $Q(1, 2, -1)$, $R(-2, 1, 0)$

25–28 ■ Find the area of $\triangle PQR$.

25. $P(1, 0, 1)$, $Q(0, 1, 0)$, $R(2, 3, 4)$

29–34 ■ Three vectors **a**, **b**, and **c** are given. (a) Find their scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. (b) Are the vectors coplanar? If not, find the volume of the parallelepiped that they determine.

33. $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{c} = 6\mathbf{i}$

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37. **Order of Operations in the Triple Product** Given three vectors **u**, **v**, and **w**, their scalar triple product can be performed in six different orders:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}),\quad \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}),\quad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}),$$
$$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}),\quad \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}),\quad \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$$

(a) Calculate each of these six triple products for the vectors:

$$\mathbf{u} = \langle 0, 1, 1 \rangle,\quad \mathbf{v} = \langle 1, 0, 1 \rangle,\quad \mathbf{w} = \langle 1, 1, 0 \rangle$$

(b) On the basis of your observations in part (a), make a conjecture about the relationships between these six triple products.

(c) Prove the conjecture you made in part (b).

9.6

CONCEPTS

1. A line in space is described algebraically by using ________ equations. The line that passes through the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$ is described by the equations $x =$ ________, $y =$ ________, $z =$ ________.

2. The plane containing the point $P(x_0, y_0, z_0)$ and having the normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described algebraically by the equation ________.

3–8 ■ Find parametric equations for the line that passes through the point P and is parallel to the vector $\mathbf{v}$.

7. $P(1, 1, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

15–20 ■ A plane has normal vector $\mathbf{n}$ and passes through the point P. (a) Find an equation for the plane. (b) Find the intercepts and sketch a graph of the plane.

15. $\mathbf{n} = \langle 1, 1, -1 \rangle$, $P(0, 2, -3)$

27–30 ■ A description of a line is given. Find parametric equations for the line.

27. The line crosses the x-axis where $x = -2$ and crosses the z-axis where $z = 10$.

29. The line perpendicular to the xz-plane that contains the point $(2, -1, 5)$.

31–34 ■ A description of a plane is given. Find an equation for the plane.

32. The plane that crosses the x-axis where $x = 1$, the y-axis where $y = 3$, and the z-axis where $z = 4$.

33. The plane that is parallel to the plane $x - 2y + 4z = 6$ and contains the origin.

34. The plane that contains the line $x = 1 - t$, $y = 2 + t$, $z = -3t$ and the point $P(2, 0, -6)$. [*Hint:* A vector from any point on the line to P will lie in the plane.]

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35. **Intersection of a Line and a Plane** A line has parametric equations

$$x = 2 + t, \qquad y = 3t, \qquad z = 5 - t$$

and a plane has equation $5x - 2y - 2z = 1$.

(a) For what value of t does the corresponding point on the line intersect the plane?

(b) At what point do the line and the plane intersect?

36. **Lines and Planes** A line is parallel to the vector $\mathbf{v}$, and a plane has normal vector $\mathbf{n}$.

(a) If the line is perpendicular to the plane, what is the relationship between $\mathbf{v}$ and $\mathbf{n}$ (parallel or perpendicular)?

(b) If the line is parallel to the plane (that is, the line and the plane do not intersect), what is the relationship between $\mathbf{v}$ and $\mathbf{n}$ (parallel or perpendicular)?

(c) Parametric equations for two lines are given. Which line is parallel to the plane $x - y + 4z = 6$? Which line is perpendicular to this plane?

Line 1: $x = 2t, \qquad y = 3 - 2t, \qquad z = 4 + 8t$

Line 2: $x = -2t, \qquad y = 5 + 2t, \qquad z = 3 + t$