3.1 Polynomial Behavior
(P.262) # 1a, 2b, 3c, 4d, 5~10, 11, 13, 21, 22, 27, 31, 34, 46, 72, 75, 82, 83
B1 B2

3.2 Dividing Polynomials
(P.270) # 3, 7, 11, 29, 43, 50, 53, 55, 56, 59, 61~66
67

3.3 Real Zeros of Polynomials   Descartes may help & upper/lower bds
(P.279) # 3, 4, 7, 10, 15, 17 (Like ex 2), 29 (by grouping)
Hints → 31, 35, 45, 58, 61, 63, 67, 70, 76, 100,
B3 Read proof of rational zero thm (pg 272)
Prove that q is a factor of a_n

3.4 Complex Numbers
(P.289) # 11, 13, 18, 25, 28, 37, 41, 43, 45, 46, 51, 53, 55, 61, 65, 71, 74~78, 79,

3.5 Fundamental Thm of Algebra   CW 56
(P.298) # $\underbrace{3, 5, 7, 9, 11, 23, 29, 30}_{\text{pattern or grouping}}$, 37, 39, $\underbrace{45, 55}_{\text{only this needs synthetic divisions}}$, 59, 65, 67.

3.6 Rational Functions
(P.312) # 7, 9, 12, 14, 15, 20, 21, 22, 25~31 odd, 45, 48, 55
58, 62, 66, Bonus 84
80, 83

4.5 (P.379) # 3, 6, 7, 8, 10, 11, 15, 21, 29, 33, 41.

Ch.4

4.1 Exponential functions
(P.336)
CW #16, 26~38 even
HW # 4, 9, 15, 17, 19~24, 25~41 odd, 42, 43, 55, 59,
65, 69, 77,
-Bring up growth doubling time same no matter what y_0 is.
-extra probs on Initial Condition exponential growth → §4.5
* double daily for a month OR \$1million?
\$1

4.2 Logarithmic Functions
(P.349)
CW #2, 4~24 even.
HW #1~39 odd. #41~46, 47, 49~63 odd, 73, 75~77
78, 83,
Bonus: 84~87. No calculator.

4.3 Laws of Logarithms
(P.356)
-CW #2~48 even. No calculator. Show steps
- $\log_6 x = \frac{\ln x}{\ln 6}$ TI-84   · TI-89 ◆ 7 log(x, 6)
HW #1~63 odd. #66~68

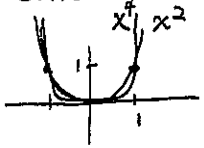
Pg 347 (ex 8)
$SNR = 10\log\left(\frac{S}{N}\right)$
3dB cutoff freq
$3 = 10\log\left(\frac{I}{I_0}\right)$
$10^{\frac{3}{10}} = \frac{S}{N}$
$S = 2N$

4.4 Preview SAT Pg 8. Eqns: ① 1-1 $\log(\square) = \log(\Delta)$, $\square = \Delta$ ② $\log_b(x) = c$, $b^c = x$ ③ $2^{\square} = $ ↓ ln
(P.367) #

3.1 Polynomials

① Pg 250 vocab

② Smooth & Continuous

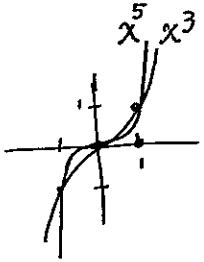


x^n

steeper $|x|>1$ grows faster when n big

flatter $|x|<1$ decreases

n even → both sides go up



n odd: 1 side up
left side down

← Limit Notation

ex1 Transformations

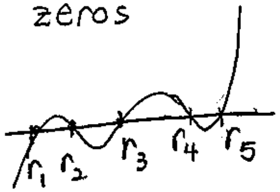
③ end behavior

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

	$a_n > 0$	$a_n < 0$
n even	$\lim_{x\to -\infty} P(x) = +\infty$	$\lim_{x\to -\infty} P(x) = -\infty$
	$\lim_{x\to +\infty} P(x) = +\infty$	$\lim_{x\to +\infty} P(x) = -\infty$
n odd	$\lim_{x\to -\infty} P(x) = -\infty$	$\lim_{x\to -\infty} P(x) = +\infty$
	$\lim_{x\to \infty} P(x) = +\infty$	$\lim_{x\to +\infty} P(x) = -\infty$

ex2 & ex3

④ zeros



ⓐ Relation of $y = P(x)$ ← function
& $P(x)=0$ ← eqn.

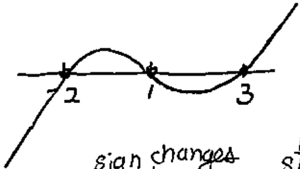
Solns are roots/zeros of function where f crosses x-axis

$$P(x) = a(x-r_1)(x-r_2)(x-r_3)(x-r_4)(x-r_5)$$

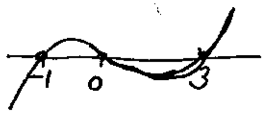
- zero = root = soln to $P(x)=0$
- $x-r$ is a factor
- r is an x-intercept (when real)

⑤ Shape of Graph near zero

ex4 $P(x) = (x+2)(x-1)(x-3)$



ex5 $P(x) = x^3 - 2x^2 - 3x$
$= x(x^2-2x-3)$
$= x(x-3)(x+1)$



Why? Multiplicity	sign changes ↓ odd: cross	left & right sign same ↓ even: tangent

$P(x) = (x+2)(x-1)^2(x-3)$

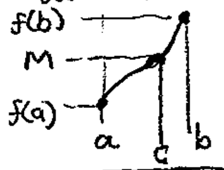
Leading term: $1x^4$
Both sides up

ex6, 7, 8

3.1

⑥ Intermediate Value Thm.

A string must cross all intermediate values



IVT

$$\boxed{\begin{array}{l} f \text{ is continuous on } [a,b]. \\ f(a) \leq M \leq f(b) \\ \Rightarrow \exists\, c \in (a,b) \text{ s.t. } f(c) = M \end{array}}$$

$\boxed{\text{ex}}$ $f(x) = x^2 - 3x + 2$

$(= (x-1)(x-2))$

Use IVT to show that f has a zero in $[0, 1.5]$

$f(0) = 2$

$f(1.5) = \frac{9}{4} - 3\cdot\frac{3}{2} + 2 = \frac{9}{4} - \frac{18}{4} + \frac{8}{4} = \frac{-1}{4}$

$f(1.5) < 0 < f(2)$

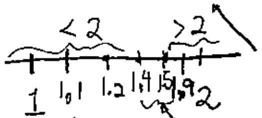
Since f is continuous, $\exists\, c \in (1.5, 2)$ s.t. $f(c) = 0$

ⓐ Estimate $\sqrt{2} \approx 1.414213562$

to within 0.005 of actual answer

Solve $\sqrt{2} = x$ $\quad \sqrt{1} = 1$ $\quad 1 < x < 2$

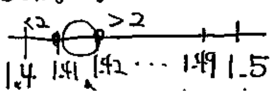
$x^2 = 2$ $\quad \sqrt{4} = 2$



Look for change from <2 to >2

x	x^2
1	
1.1	1.21
1.2	1.44
1.3	1.69
1.4	1.962

Soln in 1.4 to 1.5



Error ≤ 0.005

$\boxed{1.415 \approx \sqrt{2}}$

ⒷⒾ Use IVT to show

$f(x) = -x^3 + 2x - 1$ has a zero in $[-2, -1]$

Ⓑ2 Use IVT to estimate $\sqrt{3}$ with an error less than 0.005

Number of Local Extrema

$P(x) = a_n x^{\boxed{n}} + \dots + a_1 x + a_0$

has at most $n-1$ extrema

from Calculus

Pg 261 & SAT Pg 11

3.2 Dividing Polynomials

Dividend, Quotient, Remainder, Divisor

$$\frac{38}{7} = 5 + \frac{\boxed{2}}{7}$$

$$38 = 5 \times 7 + 2$$

$$\begin{array}{r} 5. \\ 7\overline{)38.0} \\ -35 \\ \hline 3 \end{array}$$

ⓐ $$\frac{673759}{31} \overset{?}{=} 21734 + \frac{5}{31}$$

By CALC 21734.16129...

$673759 - 21734 \times 31 = 5$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{r(x)}{D(x)}$$

$$P(x) = D(x)\,Q(x) + \underbrace{r(x)}_{\text{Leftover}}$$

ex $P(x) = 8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) - 7x + 1$

$D(x) = 2x^2 - x + 2$

OR

$$\boxed{\frac{P(x)}{D(x)} = 4x^2 + 2x + \frac{-7x+1}{2x^2 - x + 2}}$$

$$\begin{array}{rl} & 4x^2 + 2x \\ 2x^2 - x + 2 \,\big) & 8x^4 + 0x^3 + 6x^2 - 3x + 1 \\ & -(8x^4 - 4x^3 + 8x^2) \\ \hline & 4x^3 - 2x^2 - 3x \\ & 4x^3 - 2x^2 + 4x \\ \hline & -7x + 1 \end{array}$$

finished

Try SAT Pg ?

ⓐ $$\frac{x^4 - 4x^2 + 7x + 15}{x+4} = (x^3 - 4x^2 + 12x - 41) + \frac{179}{x+4}$$

$$\begin{array}{rl} & 1x^3 \quad -4x^2 + 12x - 41 \\ x\boxed{+4} \,\big) & 1x^4 + 0x^3 - 4x^2 + 7x + 15 \\ & x^4 + 4x^3 \\ \hline & -4x^3 - 4x^2 \\ & -4x^3 - 16x^2 \\ \hline & 12x^2 + 7x \\ & 12x^2 + 48x \\ \hline & -41x + 15 \\ & -41x - 164 \\ \hline & 179 \end{array}$$

Try TI-89 CALC

★ Synthetic Division - read pg 267

Only works for $D(x) = 1x - c$

$$\begin{array}{r|rrrrr} & x^4 & & & & \\ -4 & 1 & 0 & -4 & 7 & 15 \\ & & -4 & 16 & -48 & 164 \\ \hline & 1 & -4 & 12 & -41 & \boxed{179} \\ & x^3 & 2 & 1 & 0 & \end{array}$$

Remainder

ex $$\frac{2x^3 - 7x^2 + 5}{x-3} = 2x^2 - x - 3 + \frac{-4}{x-3}$$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & \boxed{-4} \\ & x^2 & x^1 & x^0 & \text{remainder} \end{array}$$

- Remainder Thm
  Factor Thm $\rightarrow$ see SAT Pg 12

*Quiz

ex4 $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$

Find Remainder of dividing by $x+2$

a) By Thm (fast) $P(-2) = -3\cdot32+5\cdot16+4\times8-14+3 = 5$

b) Synthetic (slower)

c) Long division (slowest)

b)
$$\begin{array}{r|rrrrrr} -2 & 3 & 5 & -4 & 0 & 7 & 3 \\ & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & \boxed{5} \end{array}$$

c) $x+2\,\overline{)3x^5+5x^4-4x^3+0x^2+7x+3}$

Q Use Synthetic?

$$\frac{6x^4+9x^3-3x+18}{3x^2+6} \text{ ? No}$$

$$\frac{6x^4+9x^3-3x+18}{3x+6} \text{ ? Yes} = \frac{\overbrace{2x^4+3x^3-x+6}^{P(x)}}{x+2} = (2x^3-x^2+2x-5)+\frac{16}{x+2}$$

$$\begin{array}{r|rrrrr} -2 & 2 & 3 & 0 & -1 & 6 \\ & & -4 & 2 & -4 & 10 \\ \hline & 2 & -1 & 2 & -5 & 16 \end{array}$$

Is $x+2$ a factor? No. Remainder $= P(-2)$ ✓

ex5 Factor out $x^3 - 7x + 6$. Hint: $P(1) = 0$

Soln: So $(x-1)$ is a factor

$$x^3 - 7x + 6 = (x-1)(\underbrace{x^2 + x - 6}_{Q(x)})$$
$$= (x-1)(x+3)(x-2)$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$x^2 + x - 6 = (x+3)(x-2)$

ex6 Find a poly of smallest degree w/zeros $-3, 0, 1, 5$

$P(x) = (x+3)\,x\,(x-1)(x-5) \times C$ TI-89 expand

3.3 Real zeros. (See SAT Pg 14)

(83) HW
*Bonus: read proof
Prove q is factor of a_n

Rational Zero Thm

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

- $a_0, a_1, \dots a_n \in \mathbb{Z}$ integer coefficients
- If $\frac{p}{q}$ (no common factors) is a rational zero

$\Uparrow \Downarrow$

then p is a factor of a_0 &
q is a factor of a_n (leading coeff)

Q Will it find all zeros? (Not for irrationals!) Q will it work for $f(x) = 1.9x^2 - 2.5x + 3$? No

ex2 Factor out
$P(x) = 2x^3 + x^2 - 13x + 6$. Integer coeff

Possibilities $q: \pm1, \pm2$ $p: \pm1, \pm2, \pm3, \pm6$

① $\frac{p}{q}$: $\pm1, \pm\frac{1}{2}, \pm2, \pm\cancel{\frac{2}{2}}, \pm\frac{6}{1}, \pm\frac{3}{2}, \pm\cancel{\frac{3}{1}}$

$\pm\frac{6}{2} = \pm3$

② Test until you find a zero

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 2 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array}$$

is 1 a zero $(x-1)$? NO

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

is 2 a zero? yes!

③ $P(x) = (x-2)(2x^2+5x-3)$
$= (x-2)(x+3)(2x-1)$

$\frac{1}{2}$ $\frac{3}{-1}$

$\boxed{\text{ex3}}$ $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

a) zeros

b) Sketch

q: ± 1

p: $\pm 1 \quad \pm 10$
$\pm 2 \quad \pm 5$

$\frac{p}{q} = \pm 1, \pm 2, \pm 5, \pm 10$

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline & 1 & 0 & -5 & -2 & 0 \end{array} \checkmark$$

$P(x) = (x-5)(x^3 - 5x - 2)$

q: ± 1 p: ± 1
± 2

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array} \checkmark$$

$P(x) = (x-5)(x+2)(\underbrace{x^2 - 2x - 1})$

$$x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$P(x) = (x-5)(x+2)(x-1-\sqrt{2})(x-1+\sqrt{2})$

thm <u>Descartes'</u> Rule of Signs

See SAT pg 13

$P(x) = a_n x^n + \dots + a_1 x + a_0$
ordered

sign changes or $-$even # = # positive real zeros

$P(-x)$: # sign changes or $-$even = # negative real zeros

- Finish 3.2
  - ⓐ Find remainder
    - Long division
    - synthetic
    - thm
  - ⓐ factor. Hint $P(1)=0$.
- overview HW

$\boxed{3.3}$ Real Zeros of Polynomials

$\boxed{\text{Rational Zero Thm}}$ ← see SAT packet pg.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

- real coefficients: $a_0, a_1, \dots a_n$

If $\frac{p}{q}$ (no common factors) is a rational zero,

then
- p is a factor of a_0 (const.)
- q a_n (lead coeff)

★ maybe only irrational zeros.

$\boxed{\text{ex2}}$ $P(x) = 2x^3 + x^2 - 13x + 6$

$q: \pm 1 \quad \pm 2$

$p: \pm 1 \quad \pm 6$
$\quad \pm 2 \quad \pm 3$

$\frac{p}{q}: \pm 1, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\cancel{\frac{2}{2}}, \pm 3, \pm\frac{3}{2}, \pm 6, \pm\cancel{\frac{6}{2}}$

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -13 & 6 \\ & & 2 & 2 & -10 \\ \hline & 2 & 2 & -11 & \cancel{-4} \end{array}$$

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array} \quad \text{yes}$$

→ $P(x) = (x-2)(2x^2 + 5x - 3)$

$\begin{matrix} 2 & & -1 \\ 1 & & 3 \end{matrix}$

$= \boxed{(x-2)(2x-1)(x+3)}$

★ Descartes' $\boxed{\text{Rule}}$ of Signs – see SAT

★ Upper & Lower Bounds $\boxed{\text{Thm}}$

r_1 | all ≥ 0

r_1 is an upper bound (all zeros $\leq r_1$)

r_2 | + – + – + –

– + – + – +

0 counts as ±

r_2 is lower bound (all zeros $\geq r_2$)

★ Technique to go faster ★

$\boxed{\text{ex5}}$ $P(x) = x^4 - 3x^2 + 2x - 5$

real positive zeros: 3, 2 or 1
neg

$P(-x) = x^4 - 3x^2 - 2x - 5$

Show all real zeros are in $(-3, 2)$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -3 & 2 & -5 \\ & & -3 & 9 & -18 & 48 \\ \hline & 1 & -3 & 6 & -16 & 43 \end{array}$$

alternate
-3 is a lower bound

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -3 & 2 & -5 \\ & & 2 & 4 & 2 & 8 \\ \hline & 1 & \underbrace{2 \quad 1 \quad 4 \quad +3}_{>0} \end{array}$$

2 is an upper bound

① Descartes
② $\frac{p}{q}$ possibilities
③ Test. Use bounds to guide you for rationals
④ degree is lower. Repeat

$P(x) = x^4+5x^3-5x^2-23x+10$

$\boxed{ex3}$ $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

a) factor
b) zeros
c) Sketch

- 2 or 0 real positives
- 2 or 0 real negatives

p: ±1 ±10
±2 ±5

$$\begin{array}{r|rrrrr} 2 & 1 & -5 & -5 & 23 & 10 \\ & & 2 & -6 & -22 & +2 \\ \hline & 1 & -3 & -11 & +1 & 12 \end{array}$$ ← not upper bound

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline & 1 & 0 & -5 & -2 & 0 \end{array}$$ ✓ ← 5 is a zero

← (1 more real positive zero Perhaps 10?)

$P(x) = (x-5)(x^3-5x-2)$

p: ±1 ±2

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$ ✓ ← Look for negative root

$P(x) = (x-5)(x+2)\underbrace{(x^2-2x-1)}$

$$\frac{2 \pm \sqrt{4-4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2}$$ ($2\sqrt{2}$)

$= 1 \pm \sqrt{2}$ 2.4, −0.4

$P(x) = (x-5)(x+2)(x-1+\sqrt{2})(x-1-\sqrt{2})$

$\boxed{ex6}$ $P(x) = 2x^5+5x^4-8x^3-14x^2+6x+9$

q: ±1 ±2

p: ±1 ±9
±3

- Descartes: $\boxed{2}$ or 0 positive real roots

$P(-x) = -2x^5+5x^4+8x^3-14x^2-6x+9$

1 or 3 negative real roots

- $\frac{p}{q}$: $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}, \pm 9$

$$\begin{array}{r|rrrrrr} 1 & 2 & 5 & -8 & -14 & 6 & 9 \\ & & 2 & 7 & -1 & -15 & -9 \\ \hline & 2 & 7 & -1 & -15 & -9 & 0 \end{array}$$ ✓ not upper bound

$P(x) = (x-1)(2x^4+7x^3-x^2-15x-9)$

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 7 & -1 & -15 & -9 \\ & & 3 & 15 & 21 & 9 \\ \hline & 2 & 10 & 14 & 6 & 0 \end{array}$$ ✓ upper bound

$P(x) = (x-1)(x-\frac{3}{2})(2x^3+10x^2+14x+6)$

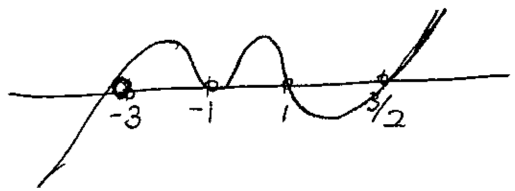
$= (x-1)(2x-3)(x^3+5x^2+7x+3)$

p: ±1, ±3

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 7 & 3 \\ & & -1 & -4 & -3 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$ ✓ factor. Not lower bound

$P(x) = (x-1)(2x-3)(x+1)\underbrace{(x^2+4x+3)}_{(x+1)(x+3)}$

Sketch



James's example. Find zeros of

(a) $P(x) = 6x^3 - 4x^2 + 3x - 2$

$P(-x) = -6x^3 - 4x^2 - 3x - 2$

Descarte: 1 or 3 positive real
0 negative real roots

$q: \pm1 \ \pm6 \ \pm2 \ \pm3$ $\qquad$ $p: \pm1 \ \pm2$

$\frac{p}{q}: \pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm1, \pm\cancel{\frac{2}{6}}, \pm\frac{2}{3}, \pm\cancel{\frac{2}{2}}, \pm2$

★ No negative roots

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

← 1 is upper bound

Look in $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$

$$\begin{array}{r|rrrr} \frac{2}{3} & 6 & -4 & 3 & -2 \\ & & 4 & 0 & 2 \\ \hline & 6 & 0 & 3 & 0 \end{array}$$

✓ $\frac{2}{3}$ is upper bound

$P(x) = (x - \frac{2}{3})(6x^2 + 3)$

$= \boxed{(3x-2)(2x^2+1)} = (3x-2)(x - i\frac{\sqrt{2}}{2})(x + i\frac{\sqrt{2}}{2})$

irreducible
other zeros are imaginary

$2x^2+1$

$x = \pm\sqrt{-\frac{1}{2}} = \pm i\frac{\sqrt{2}}{2}$

70. $P(x) = 2x^3 - 3x^2 - 8x + 12$ ← 0 or 2 positives

$P(-x) = -2x^3 - 3x^2 + 8x + 12$ ← 1 negative

$q: \pm1 \ \pm2$ $\qquad$ $p: \pm1 \ \pm12 \ \pm2 \ \pm6 \ \pm3 \ \pm4$

$\frac{p}{q}: \pm\frac{1}{2}, \pm1, \pm\cancel{\frac{2}{2}}, \pm2, \pm\frac{3}{2}, \pm3, \pm\cancel{\frac{4}{2}}, \pm\cancel{\frac{6}{2}}, \pm4, \pm6, \pm\cancel{\frac{12}{2}}, \pm12$

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -8 & 12 \\ & & -2 & 5 & 3 \\ \hline & 2 & -5 & -3 & 15 \end{array}$$

← not lower bound
Look Lower

$$\begin{array}{r|rrrr} -6 & 2 & -3 & -8 & 12 \\ & & -12 & 75 & \\ \hline & 2 & -15 & 67 & - \end{array}$$

← Lower bound

$$\begin{array}{r|rrrr} -3 & 2 & -3 & -8 & 12 \\ & & -6 & 27 & -57 \\ \hline & 2 & -9 & 19 & - \end{array}$$

← lower bound

$$\begin{array}{r|rrrr} -2 & 2 & -3 & -8 & 12 \\ & & -4 & 14 & -12 \\ \hline & 2 & -7 & 6 & 0 \end{array}$$

✓ Lower bound

$P(x) = (x+2)(2x^2 - 7x + 6) = \boxed{(x+2)(x-2)(2x-3)}$

$\begin{matrix} 1 & \times & -2 \\ 2 & & -3 \end{matrix}$

1000 is an upper bound..

$\boxed{3.4}$ Complex Numbers

standard form

$z = a + bi$

$z \in \mathbb{C} \qquad a, b \in \mathbb{R}$

$Re\{z\} = a$ real part

$Im\{z\} = b$ imaginary part

$\bar{z} = a - bi$ complex conjugate

$x^2 + 4 = 0$

no real soln.

$x = \pm\sqrt{-4}$

$= \pm\sqrt{4}\sqrt{-1}$

$= \pm 2i$

$\boxed{\begin{array}{l} i^2 = -1 \\ i = \sqrt{-1} \end{array}}$

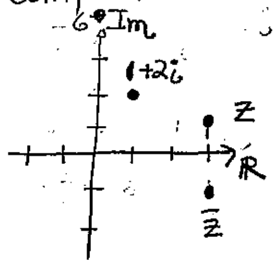
- elegant soln to polynomials
- phase

$\boxed{ex1}$

z	$Re\{z\}$	$Im\{z\}$	$\bar{z}$
$\frac{1}{2} - \frac{2}{3}i$	$\frac{1}{2}$	$-\frac{2}{3}$	$\frac{1}{2} + \frac{2}{3}i$
$6i$	0	6	$0 - 6i$

Ch. 8

Complex Plane



Arithmetic of Complex Numbers

$z_1 = a + bi$

$z_2 = c + di$

$z_1 \pm z_2 = (a \pm c) + (b \pm d)i$

$z_1 z_2 = (ac - bd) + (ad + bc)i$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$\boxed{ex2}$ $z_1 = 3 + 5i$

$z_2 = 4 - 2i$

a) $z_1 + z_2 = 7 + 3i$

b) $z_1 - z_2 = -1 + 7i$

c) $z_1 z_2 = (12 + 10) + (-6 + 20)i$

$= 22 + 14i$

Cycles of 4

$i = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = i^2 i^2 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$i^{21} = i^{20} i^1$

$i^{103} = i^{100+3} = i^3 = -i$

$i^{28} = 1$

$i^{33} = i^{32+1} = i$

$i^{38} = i^{36+2} = i^2 = -1$

$\boxed{ex3}$

a) $$\frac{z_1}{z_2} = \frac{3+5i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{(12-10)+(6+20)i}{16+4}$$

$$= \frac{2+26i}{20} = \frac{1}{10} + \frac{13}{10}i$$ Show CALC TI-89

b) $$\frac{7+3i}{4i}\left(\frac{-4i}{-4i}\right) = \frac{12-28i}{16} = \frac{3}{4} - \frac{7}{4}i$$

$\boxed{ex}$ factor out

complex roots must come in conjugate pairs

$x^2 + 4 = 0$

$x = \pm 2i$

$1 + 3i$ root

$1 - 3i$ root too

Q poly $= (x - (1+3i))(x - (1-3i))$

$= ((x-1) - 3i)((x-1) + 3i)$

$(A \quad - B)(A + B) = A^2 - B^2$

$= (x^2 - 2x + 1) + 9$

Q write poly of smallest degree with zero: $2 - 5i$

$((x-2) + 5i)((x-2) - 5i)$

$= (x^2 - 4x + 4) + 25$

Q write poly of smallest degree with zero: $3 + 2i$

$((x-3) - 2i)((x-3) + 2i) = (x^2 - 6x + 9 + 4)$

Q Poly of smallest degree with zero: $\pi - ei$

$((x-\pi) + ei)((x-\pi) - ei)$

$= x^2 - 2\pi x + \pi^2 + e^2$

$\boxed{\text{ex4}}$

a) $\sqrt{-1} = \sqrt{1}\sqrt{-1} = i$

c) $\sqrt{-3} = \sqrt{3}\sqrt{-1} = i\sqrt{3}$

b) $\sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$

Careful: $\sqrt{6} = \boxed{\sqrt{2}}\sqrt{\boxed{3}}$ ← positive OK

NOT true when both negative!

$$\underbrace{\sqrt{-2}\ \sqrt{-3}} \neq \sqrt{6}$$

$i\sqrt{2}\ \ i\sqrt{3}$ ← correct way: $i\sqrt{r}$ first

Then multiply

$i^2\sqrt{6}$

$-\sqrt{6}$

$\boxed{\text{ex5}}$ express in standard form

$$(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$$

$$= (2\sqrt{3} - i\sqrt{3})(3 + 2i)$$

$$= (6\sqrt{3} + 2\sqrt{3}) + (4\sqrt{3} - 3\sqrt{3})i$$

$$= 8\sqrt{3} + i\sqrt{3}$$

$\boxed{\text{ex7}}$ factor $4x^2 - 24x + 37 = 0$

$$x = \frac{24 \pm \sqrt{24^2 - 4(4)(37)}}{2(4)} = \frac{24 \pm \sqrt{-16}}{8}$$

$$= \frac{24 \pm 4i}{8} = \underbrace{3 \pm \tfrac{1}{2}i}_{\text{conjugate pair}}$$

$\boxed{3.5}$ Fundamental Thm of Algebra

See SAT pg...

$\boxed{\text{ex1}}$ Factor

$$P(x) = x^3 - 3x^2 + x - 3$$

$$= x^2(x - 3) + (x - 3)$$

$$= (x-3)(x^2+1) = (x-3)(x-i)(x+i)$$

simpler: By grouping

$$(x^2 + c) = (x - i\sqrt{c})(x + i\sqrt{c})$$

$$(x^2 - (i\sqrt{c})^2)$$

$$(x^2 + e) = (x - i\sqrt{e})(x + i\sqrt{e})$$

$$(x^2 + 25) = (x - 5i)(x + 5i)$$

$\boxed{\text{ex2}}$ $P(x) = x^3 - 2x + 4$

$P(-x) = -x^3 + 2x + 4$

Descarte: 0 or 2 positives

1 negative real

p: ± 1 ± 4 ± 2

-2	1	0	-2	4
		-2	4	-4
	1	-2	2	0 ✓

$x^2 - 2x + 2$

$$x = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

$$P(x) = \boxed{(x+2)(x-1-i)(x-1+i)}$$

• Multiplicity Counts

$\boxed{\text{ex3}}$ $P(x) = 3x^5 + 24x^3 + 48x$

$$= 3x(x^4 + 8x^2 + 16)$$

$$= 3x(x^2 + 4)^2$$

$$= 3x(x - 2i)^2(x + 2i)^2$$

ex4 degree 4 . zeros: i, $-i$, 2, -2 . $P(3)=25$

a) $P(x) = \frac{1}{2}(x-i)(x+i)(x-2)(x+2)$

$\frac{1}{2}(x^2+1)(x^2-4)$ $\quad x^4-3x^2-4$

$25 = \frac{1}{2}(10\times 5)$

$P(x) = \frac{1}{2}x^4 - \frac{3}{2}x^2 - 2$

b) $Q(x)$ degree 4
zeros: $-2, 0$
multiplicity 3

$Q(x) = (x+2)^3 x$

$= (x^3 + 3x^2 2 + 3x 2^2 + 2^3) x$

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{matrix}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$Q(x) = a(x^4 + 6x^3 + 12x^2 + 8x)$

$\boxed{\text{Conjugate zeros Thm}}$ ①

$P(z)=0 \Rightarrow P(\bar{z})=0$

- Use the fact that $(\bar{z})^n = \overline{z^n}$

$(re^{i\theta})^n = r^n e^{-in\theta} = (r^n e^{in\theta})^* = \overline{z^n}$

- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\quad \overline{(a+bi)}\,\overline{(c+di)} = (a-bi)(c-di)$

$\overline{(a+bi)(c+di)} = \overline{(ac-bd) + (ad+bc)i}$

$= (ac-bd) - (ad+bc)i$ $\quad \stackrel{\checkmark}{=}$

Bonus Later, Cardano

Show $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = 4$

$(2+11i)^{\frac{1}{3}} + (2-11i)^{\frac{1}{3}}$

$= z^{\frac{1}{3}} + (\bar{z})^{\frac{1}{3}} = z^{\frac{1}{3}} + \overline{(z^{\frac{1}{3}})}$

$= \mathrm{Re}\{z^{\frac{1}{3}}\}\cdot\frac{1}{2} = \frac{1}{2}\cdot r^{\frac{1}{3}}\cos\left(\frac{\theta+2\pi k}{3}\right)$

$z = 2+11i$

$= re^{i\theta}$ $\quad r=\sqrt{125}$ $\quad \tan\theta = \frac{11}{2}$

$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{i(\theta+2\pi k)/3}$

$k=0$
1
2

② So any polynomial w/ real coeffs can be factored into linear or irreducible quadratics with all real coefficients

$\therefore P(x) = (x-\underbrace{r_1}_{\text{real}})\underbrace{(x-a+bi)(x-a-bi)}_{\text{irreducible quadratic}}\ldots$

Proof:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

$P(z) = 0$

$P(\bar{z}) \stackrel{?}{=} a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \ldots + a_1\bar{z} + a_0$

$= \bar{a}_n(\overline{z^n}) + \bar{a}_{n-1}(\overline{z^{n-1}}) + \ldots + \bar{a}_1\bar{z} + \bar{a}_0$

$= \overline{a_n z^n + \ldots + a_1 z + a_0}$

$= \overline{P(z)} = \overline{0} = 0$

$\boxed{\text{ex6}}$ Polynomial with a Specified Complex Zero

$P(x)$ has degree 3

zeros $\frac{1}{2}$ and $3-i$ $\quad 3+i$

$$P(x) = (x-\tfrac{1}{2})(x-3+i)(x-3-i) \quad \times a$$
$$= (x-\tfrac{1}{2})(x^2-6x+9+1) \quad \times a$$
$$= (x-\tfrac{1}{2})(x^2-6x+10) \quad \times a$$
$$= x^3 - 6.5x^2 + 13x - 5 \quad \times a$$

Get integer coeffs. $\frac{13}{2}$ $\quad a = 2$

$$P(x) = 2x^3 - 13x^2 + 26x - 10$$

$\boxed{\text{ex7}}$ more Descartes' : SAT

$\boxed{\text{ex8}}$ Factor $P(x) = x^4 + 2x^2 - 8$.

a) into Linear & irreducible quadratic factors with real coeffs.

b) Completely factor into linear factors with complex coeffs.

Soln.

a) $x^4 + 2x^2 - 8 = (x^2+4)(x^2-2)$

$\begin{matrix} 1 & & 4 \\ 1 & & -2 \end{matrix}$ $\quad = (x^2+4)(x-\sqrt{2})(x+\sqrt{2})$

irreducible
$D<0$

b) $P(x) = (x-2i)(x+2i)(x-\sqrt{2})(x+\sqrt{2})$

$\boxed{3.6}$ Rational Functions

① See SAT Pg 14. & domain

$$f(x) = \frac{P(x)}{Q(x)} \leftarrow \text{polynomials}$$

② Simplest : Rectangular Hyperbola

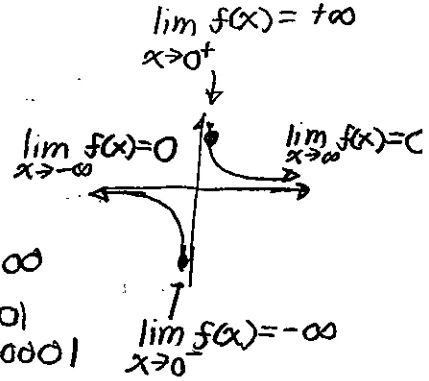
$\boxed{\text{ex1}}$ Graph $f(x) = \frac{1}{x}$

$x = 0.001 \Rightarrow f(x) = 1000$

$x = -0.0001 \Rightarrow f(x) = -10000$

$x = 10000 \Rightarrow f(x) = 0.0001$

$x = -10000 \Rightarrow f(x) = -0.0001$

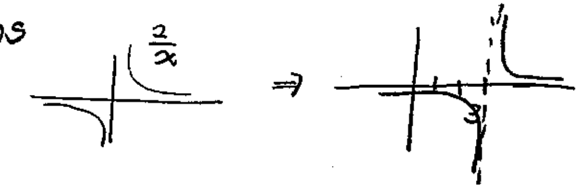


$\boxed{\text{ex2}}$ Using Transformations

a) $r(x) = \dfrac{2}{x-3}$

$= 2r(x-3)$

steeper $\quad$ Right



Horiz. Asymptote: $y=0$

Vertical : $x=3$

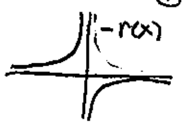
b) $f(x) = \dfrac{3x+5}{x+2} \longleftarrow$ Get in proper form

$\begin{array}{r|rr} -2 & 3 & 5 \\ & & -6 \\ \hline & 3 & -1 \end{array}$

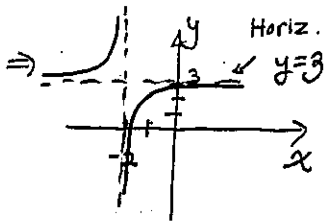
$= 3 - \dfrac{1}{x+2}$

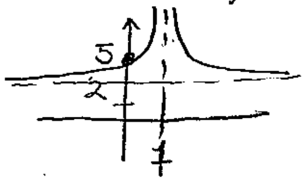
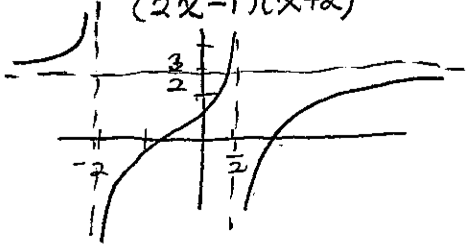
$= 3 - r(x+2)$

② ① $\quad$ ①′



vertical : $x=-2$



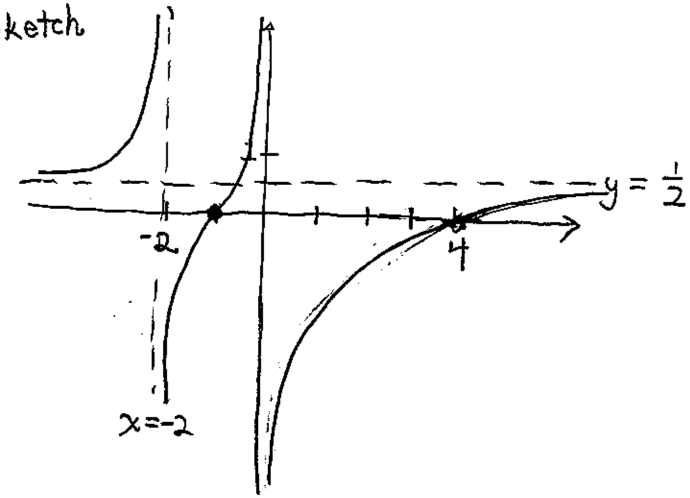
$\boxed{\text{examples}}$	Domain	vertical	Horiz	slant
$\boxed{ex3}$ $r(x) = \frac{2x^2-4x+5}{x^2-2x+1} = \frac{2x^2-4x+5}{(x-1)^2}$ 	$x \neq 1$	$x=1$ $\lim_{x\to 1^-, 1^+} r(x) = +\infty$	$y=2$ $\lim_{x\to\pm\infty} \frac{x^2(2-\frac{4}{x}+\frac{5}{x^2})}{x^2(1-\frac{2}{x}+\frac{1}{x^2})} = 2$	none
$\boxed{ex4}$ $r(x) = \frac{3x^2-2x-1}{2x^2+3x-2} = \frac{3x^2-2x-1}{(2x-1)(x+2)}$ 	$x \neq +\frac{1}{2}, -2$	$x=\frac{1}{2}$ $x=-2$	$y=\frac{3}{2}$ $\lim_{x\to\pm\infty} \frac{x^2(3-\frac{2}{x}-\frac{1}{x^2})}{x^2(2+\frac{3}{x}-\frac{2}{x^2})} = 3$	none
$G(x) = \frac{x+1}{x-2}$	$x \neq 2$	$x=2$	$y=1$	none
$f(x) = \frac{2x+3}{x^2+1}$	$\mathbb{R}$	none	$y=0$	none
$g(x) = \frac{4x^2+1}{3x^2}$	$x \neq 0$	$x=0$	$y=\frac{4}{3}$	none
$h(x) = \frac{x^3+1}{x-2}$	$x \neq 2$	$x=2$	none	none
$\frac{(x-1)(2x+1)\cancel{(x-3)}}{\cancel{(x-3)}^2(x+2)^2}$	$x \neq 3, -2$	$x=-2$	$y=2$	none

- Go to SAT Pg 16
- $\boxed{ex7}$ Graph $r(x) = \frac{(x^2-3x-4)\cancel{(x-3)}}{(2x^2+4x)\cancel{(x-3)}} = \frac{(x-4)(x+1)}{2x(x+2)}$

1. Domain $x \neq -2, 0, 3$
2. Vertical Asymp: $x=0$, $x=-2$
   Horiz. Asymp: $y=\frac{1}{2}$
3. Same-Sign Intervals

x		-2		-1		0		4	
$\frac{(x-4)(x+1)}{2x(x+2)}$	+	$\pm\infty$	−	0	+	$\pm\infty$	−	0	+

4. x-intercept: $x=-1, 4$
   y-intercept: $\frac{-4}{0}$ none
5. Sketch



$$\frac{-2(x+5)(x+3)^2}{(2x-1)(x+1)}$$

$$\frac{-2(x+5)^2\cancel{(2x-1)}(x+3)}{\cancel{(x+5)}\cancel{(2x-1)}^2(x+1)^1}$$

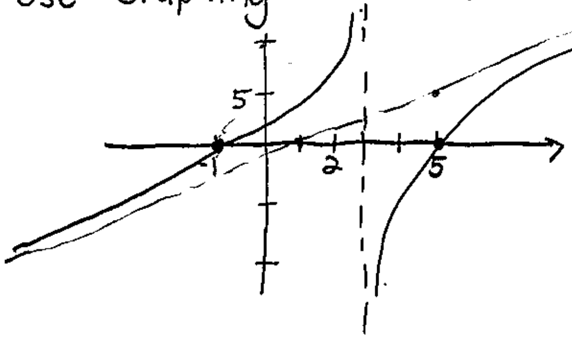
Domain	Vertical	Horiz
$x \neq -5, -1, \frac{1}{2}$	$x=-1$ $x=\frac{1}{2}$	$y=-1$

Slant asymptote

$\boxed{ex8}$ $r(x) = \dfrac{x^2-4x-5}{x-3} = (x-1) - \dfrac{8}{x-3}$

Use Graphing Calculator

Guess? $y = x-1$



SAT Pg 15

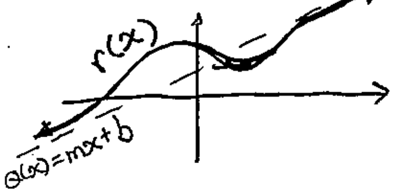
$$r(x) = \frac{P(x)}{D(x)} = \underbrace{Q(x)}_{\text{line}} + \frac{r(x)}{D(x)}$$

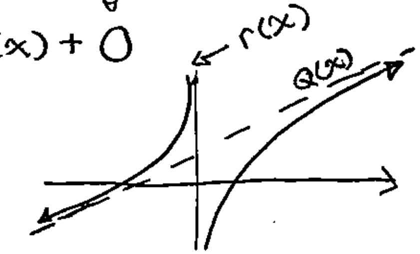
$P(x)$ ← 1 more than denominator

$r(x)$ ← 1 less

When x is huge:

$r(x) \approx Q(x) + 0$





$$\begin{array}{r|l} & x-1 \\ \hline x-3 & x^2-4x-5 \\ & \underline{x^2-3x} \\ & -x-5 \\ & \underline{-x+3} \\ & -8 \end{array}$$

Pg 15 $\boxed{ex}$ Slant

Pg 16 $\boxed{ex}$ Slant