1.1 EXERCISES

CONCEPTS

1. Give an example of each of the following:
   (a) A natural number
   (b) An integer that is not a natural number
   (c) A rational number that is not an integer
   (d) An irrational number

2. Complete each statement and name the property of real numbers you have used.
   (a) $ab =$ ______; ______ Property
   (b) $a + (b + c) =$ ______; ______ Property
   (c) $a(b + c) =$ ______; ______ Property

3. The set of numbers between but not including 2 and 7 can be written as follows:
   ______ in set-builder notation and
   ______ in interval notation.

4. The symbol $|x|$ stands for the ______ of the number x. If x is not 0, then the sign of $|x|$ is always ______.

SKILLS

5–6 ■ List the elements of the given set that are
   (a) natural numbers
   (b) integers
   (c) rational numbers
   (d) irrational numbers

5. $\{1.001, 0.333\ldots, -\pi, -11, 11, \frac{13}{15}, \sqrt{16}, 3.14, \frac{15}{3}\}$
6. $\{0, -10, 50, \frac{22}{7}, 0.538, \sqrt{7}, 1.2\overline{3}, -\frac{1}{3}, \sqrt[3]{2}\}$

7–14 ■ State the property of real numbers being used.

7. $2(3 + 5) = (3 + 5)2$
8. $7 + 10 = 10 + 7$
9. $(x + 2y) + 3z = x + (2y + 3z)$
10. $2(A + B) = 2A + 2B$
11. $(5x + 1)3 = 15x + 3$
12. $(x + a)(x + b) = (x + a)x + (x + a)b$
13. $7(a + b + c) = 7(a + b) + 7c$
14. $2x(3 + y) = (3 + y)2x$

15–18 ■ Rewrite the expression using the given property of real numbers.

15. Commutative Property of addition, $x + 3 =$
16. Associative Property of multiplication, $7(3x) =$
17. Distributive Property, $5x + 5y =$
18. Distributive Property, $4(A + B) =$

19–24 ■ Use properties of real numbers to write the expression without parentheses.

19. $(a - b)8$ 20. $3(x + y)$
21. $4(2m)$ 22. $\frac{4}{3}(-6y)$
23. $-\frac{5}{2}(2x - 4y)$ 24. $(3a)(b + c - 2d)$

25–30 ■ Perform the indicated operations.

25. (a) $\frac{3}{10} + \frac{4}{15}$ (b) $\frac{1}{4} + \frac{1}{5}$
26. (a) $\frac{2}{3} - \frac{3}{5}$ (b) $1 + \frac{5}{8} - \frac{1}{6}$
27. (a) $(3 + \frac{1}{4})(1 - \frac{4}{5})$ (b) $(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} + \frac{1}{3})$
28. (a) $\frac{2}{3}(6 - \frac{3}{2})$ (b) $0.25(\frac{8}{9} + \frac{1}{2})$
29. (a) $\dfrac{2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{3}}$ (b) $\dfrac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$
30. (a) $\dfrac{2}{\frac{2}{3}} - \dfrac{\frac{2}{3}}{2}$ (b) $\dfrac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{9}}$

31–32 ■ Place the correct symbol (<, >, or =) in the space.

31. (a) $\frac{2}{3}$ ☐ 0.67 (b) $\frac{2}{3}$ ☐ -0.67 (c) $|0.67|$ ☐ $|-0.67|$
32. (a) 3 ☐ $\frac{7}{2}$ (b) -3 ☐ $-\frac{7}{2}$ (c) 3.5 ☐ $\frac{7}{2}$

33–36 ■ State whether each inequality is true or false.

33. (a) $\dfrac{10}{11} < \dfrac{12}{13}$ (b) $-\dfrac{1}{2} < -1$
34. (a) $-6 < -10$ (b) $\sqrt{2} > 1.41$
35. (a) $-\pi > -3$ (b) $8 \le 9$
36. (a) $1.1 > 1.\overline{1}$ (b) $8 \le 8$

37–38 ■ Write each statement in terms of inequalities.

37. (a) y is negative
   (b) z is greater than 1
   (c) b is at most 8
   (d) w is positive and is less than or equal to 17
   (e) y is at least 2 units from π
38. (a) x is positive
   (b) t is less than 4
   (c) a is greater than or equal to π
   (d) x is less than $\frac{1}{3}$ and is greater than -5
   (e) The distance from p to 3 is at most 5

39–42 ■ Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \quad B = \{2, 4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

39. (a) $A \cup B$ (b) $A \cap B$
40. (a) $B \cup C$ (b) $B \cap C$
41. (a) $A \cup B \cup C$ (b) $A \cap B \cap C$
42. (a) $A \cup C$ (b) $A \cap C$

43–44 ■ Find the indicated set if

$$A = \{x \mid x \ge -2\} \quad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \le 5\}$$

43. (a) $B \cup C$ (b) $B \cap C$
44. (a) $A \cap C$ (b) $A \cap B$

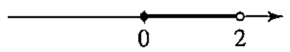
45–50 ■ Express the interval in terms of inequalities, and then graph the interval.

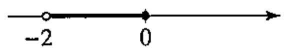
45. $(-3, 0)$ 46. $(2, 8]$
47. $[-6, -\frac{1}{2}]$ 48. $[2, 8)$
49. $(-\infty, 1)$ 50. $[2, \infty)$

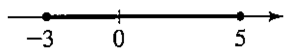
51–56 ■ Express the inequality in interval notation, and then graph the corresponding interval.

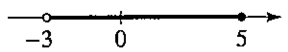
51. $x \le 1$ 52. $1 \le x \le 2$
53. $x \ge -5$ 54. $-2 < x \le 1$
55. $x > -1$ 56. $-5 < x < 2$

57–58 ■ Express each set in interval notation.

57. (a) 

(b) 

58. (a) 

(b) 

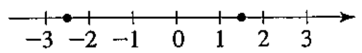
59–64 ■ Graph the set.

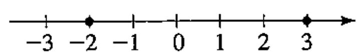
59. $(-2, 0) \cup (-1, 1)$ 60. $(-2, 0) \cap (-1, 1)$
61. $[-4, 6) \cup [0, 8)$ 62. $[-4, 6] \cap [0, 8)$
63. $(-\infty, 6] \cap (2, 10)$ 64. $(-\infty, -4) \cup (4, \infty)$

65–70 ■ Evaluate each expression.

65. (a) $|100|$ (b) $|-73|$
66. (a) $|\sqrt{5} - 5|$ (b) $|10 - \pi|$
67. (a) $\big|2 - |-12|\big|$ (b) $-1 - \big|1 - |-1|\big|$
68. (a) $\big||-6| - |-4|\big|$ (b) $\dfrac{-1}{|-1|}$
69. (a) $\left|\dfrac{-6}{24}\right|$ (b) $\left|\dfrac{7 - 12}{12 - 7}\right|$
70. (a) $|(-2) \cdot 6|$ (b) $|(-\frac{1}{3})(-15)|$

71–74 ■ Find the distance between the given numbers.

71. 

72. 

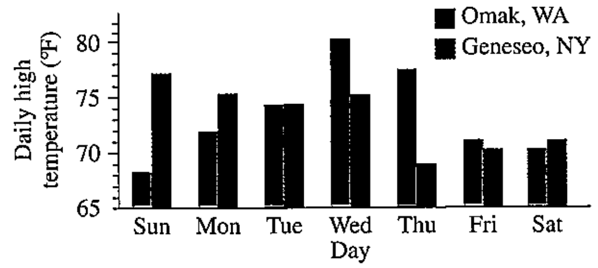
73. (a) 2 and 17 (b) -3 and 21 (c) $\frac{11}{8}$ and $-\frac{3}{10}$
74. (a) $\frac{7}{15}$ and $-\frac{1}{21}$ (b) -38 and -57 (c) -2.6 and -1.8

75–76 ■ Express each repeating decimal as a fraction. (See the margin note on page 2.)

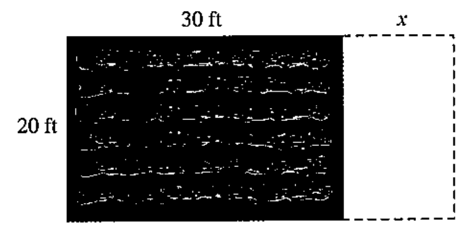
75. (a) $5.\overline{23}$ (b) $1.3\overline{7}$ (c) $2.\overline{135}$
76. (a) $0.\overline{7}$ (b) $0.2\overline{8}$ (c) $0.\overline{57}$

APPLICATIONS

77. **Temperature Variation** The bar graph shows the daily high temperatures for Omak, Washington, and Geneseo, New York, during a certain week in June. Let T_O represent the temperature in Omak and T_G the temperature in Geneseo. Calculate $T_O - T_G$ and $|T_O - T_G|$ for each day shown. Which of these two values gives more information?



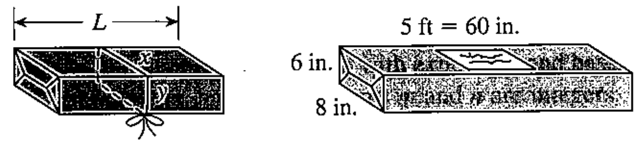
78. **Area of a Garden** Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600\ \text{ft}^2$. She decides to make it longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?



79. **Mailing a Package** The post office will only accept packages for which the length plus the "girth" (distance around) is no more than 108 inches. Thus, for the package in the figure, we must have

$$L + 2(x + y) \le 108$$

(a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?
(b) What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in?



DISCOVERY ■ DISCUSSION ■ WRITING

80. Signs of Numbers Let a, b, and c be real numbers such that $a > 0$, $b < 0$, and $c < 0$. Find the sign of each expression.

(a) $-a$ (b) $-b$ (c) bc
(d) $a - b$ (e) $c - a$ (f) $a + bc$
(g) $ab + ac$ (h) $-abc$ (i) ab^2

81. Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?

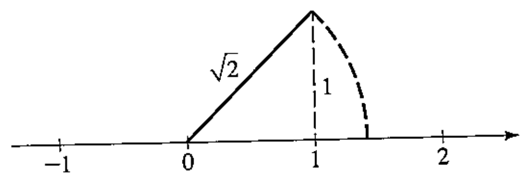
82. Combining Rational Numbers with Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? In general, what can you say about the sum of a rational and an irrational number? What about the product?

83. Limiting Behavior of Reciprocals Complete the tables. What happens to the size of the fraction $1/x$ as x gets large? As x gets small?

x	$1/x$
1	
2	
10	
100	
1000	

x	$1/x$
1.0	
0.5	
0.1	
0.01	
0.001	

84. Irrational Numbers and Geometry Using the following figure, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? What about $\sqrt{6}$? List some other irrational numbers that can be located this way.



85. Commutative and Noncommutative Operations We have seen that addition and multiplication are both commutative operations.

(a) Is subtraction commutative?

(b) Is division of nonzero real numbers commutative?

1.2 Exponents and Radicals

Integer Exponents ▶ Rules for Working with Exponents ▶ Scientific Notation ▶ Radicals ▶ Rational Exponents ▶ Rationalizing the Denominator

In this section we give meaning to expressions such as $a^{m/n}$ in which the exponent m/n is a rational number. To do this, we need to recall some facts about integer exponents, radicals, and nth roots.

▼ Integer Exponents

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3. In general, we have the following definition.

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the **nth power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

EXAMPLE 1 | Exponential Notation

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3)\cdot(-3)\cdot(-3)\cdot(-3) = 81$

(c) $-3^4 = -(3\cdot3\cdot3\cdot3) = -81$

NOW TRY EXERCISE 15 ■

Note the distinction between $(-3)^4$ and -3^4. In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2:

$$5^4 \cdot 5^2 = \underbrace{(5\cdot5\cdot5\cdot5)}_{4 \text{ factors}}\underbrace{(5\cdot5)}_{2 \text{ factors}} = \underbrace{5\cdot5\cdot5\cdot5\cdot5\cdot5}_{6 \text{ factors}} = 5^6 = 5^{4+2}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n, we have

$$a^m a^n = \underbrace{(a\cdot a\cdot\cdots\cdot a)}_{m \text{ factors}}\underbrace{(a\cdot a\cdot\cdots\cdot a)}_{n \text{ factors}} = \underbrace{a\cdot a\cdot a\cdot\cdots\cdot a}_{m+n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2 | Zero and Negative Exponents

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$

(c) $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

NOW TRY EXERCISE 17 ■

▼ Rules for Working with Exponents

Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases a and b are real numbers, and the exponents m and n are integers.

DIOPHANTUS lived in Alexandria about 250 A.D. His book *Arithmetica* is considered the first book on algebra. In it he gives methods for finding integer solutions of algebraic equations. *Arithmetica* was read and studied for more than a thousand years. Fermat (see page 99) made some of his most important discoveries while studying this book. Diophantus' major contribution is the use of symbols to stand for the unknowns in a problem. Although his symbolism is not as simple as what we use today, it was a major advance over writing everything in words. In Diophantus' notation the equation

$$x^5 - 7x^2 + 8x - 5 = 24$$

is written

$$\Delta K^{\upsilon}\alpha\varsigma\eta \pitchfork \Delta^{\upsilon}\zeta\mathring{M}\varepsilon\iota^{\sigma}\kappa\delta$$

Our modern algebraic notation did not come into common use until the 17th century.

EXAMPLE 11 | Using the Laws of Exponents with Rational Exponents

(a) $a^{1/3}a^{7/3} = a^{8/3}$ — Law 1: $a^m a^n = a^{m+n}$

(b) $\dfrac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$ — Law 1, Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(c) $(2a^3b^4)^{3/2} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2}$ — Law 4: $(abc)^n = a^n b^n c^n$

$= (\sqrt{2})^3 a^{3(3/2)} b^{4(3/2)}$ — Law 3: $(a^m)^n = a^{mn}$

$= 2\sqrt{2}a^{9/2}b^6$

(d) $\left(\dfrac{2x^{3/4}}{y^{1/3}}\right)^3\left(\dfrac{y^4}{x^{-1/2}}\right) = \dfrac{2^3(x^{3/4})^3}{(y^{1/3})^3}\cdot(y^4x^{1/2})$ — Laws 5, 4, and 7

$= \dfrac{8x^{9/4}}{y}\cdot y^4x^{1/2}$ — Law 3

$= 8x^{11/4}y^3$ — Laws 1 and 2

NOW TRY EXERCISES **61**, **63**, **67**, AND **69** ■

EXAMPLE 12 | Simplifying by Writing Radicals as Rational Exponents

(a) $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3})$ — Definition of rational exponents

$= 6x^{1/2+1/3} = 6x^{5/6}$ — Law 1

(b) $\sqrt{x\sqrt{x}} = (xx^{1/2})^{1/2}$ — Definition of rational exponents

$= (x^{3/2})^{1/2}$ — Law 1

$= x^{3/4}$ — Law 3

NOW TRY EXERCISES **71** AND **75** ■

▼ Rationalizing the Denominator

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\cdot 1 = \frac{1}{\sqrt{a}}\cdot\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m}\sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

EXAMPLE 13 | Rationalizing Denominators

This equals 1

(a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

(b) $\dfrac{1}{\sqrt[3]{x^2}} = \dfrac{1}{\sqrt[3]{x^2}}\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}} = \dfrac{\sqrt[3]{x}}{\sqrt[3]{x^3}} = \dfrac{\sqrt[3]{x}}{x}$

(c) $\sqrt[7]{\dfrac{1}{a^2}} = \dfrac{1}{\sqrt[7]{a^2}} = \dfrac{1}{\sqrt[7]{a^2}}\dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}} = \dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^7}} = \dfrac{\sqrt[7]{a^5}}{a}$

NOW TRY EXERCISES **89** AND **91** ■

1.2 EXERCISES

CONCEPTS

1. (a) Using exponential notation, we can write the product $5\cdot5\cdot5\cdot5\cdot5\cdot5$ as ______.

(b) In the expression 3^4, the number 3 is called the ______, and the number 4 is called the ______.

2. (a) When we multiply two powers with the same base, we ______ the exponents. So $3^4\cdot3^5 =$ ______.

(b) When we divide two powers with the same base, we ______ the exponents. So $\dfrac{3^5}{3^2} =$ ______.

3. (a) Using exponential notation, we can write $\sqrt[3]{5}$ as ______.

(b) Using radicals, we can write $5^{1/2}$ as ______.

(c) Is there a difference between $\sqrt{5^2}$ and $(\sqrt{5})^2$? Explain.

4. Explain what $4^{3/2}$ means, then calculate $4^{3/2}$ in two different ways:

$(4^{1/2})^{\ } =$ ______ or $(4^3)^{\ } =$ ______

5. Explain how we rationalize a denominator, then complete the following steps to rationalize $\dfrac{1}{\sqrt{3}}$:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\ }{\ } = \frac{\ }{\ }$$

6. Find the missing power in the following calculation: $5^{1/3}\cdot5^{\ } = 5$.

SKILLS

7–14 ■ Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
7.	$\sqrt[3]{7^2}$	
8.	$\dfrac{1}{\sqrt{5}}$	
9.		$4^{2/3}$
10.		$11^{-3/2}$
11.		$2^{-1.5}$
12.	$\sqrt[5]{5^3}$	
13.		$a^{2/5}$
14.	$\dfrac{1}{\sqrt{x^5}}$	

15–24 ■ Evaluate each expression.

15. (a) $5^4\cdot5^{-2}$ (b) $\dfrac{10^7}{10^4}$ (c) $\dfrac{3}{3^{-2}}$

16. (a) -3^2 (b) $(-3)^2$ (c) $\left(\frac{1}{3}\right)^4(-3)^2$

17. (a) $\left(\frac{5}{3}\right)^0 2^{-1}$ (b) $\dfrac{2^{-3}}{3^0}$ (c) $\left(\frac{1}{4}\right)^{-2}$

18. (a) $\left(-\frac{2}{3}\right)^{-3}$ (b) $\left(\frac{3}{2}\right)^{-2}\cdot\frac{9}{16}$ (c) $\left(\frac{1}{2}\right)^4\cdot\left(\frac{5}{2}\right)^{-2}$

19. (a) $\sqrt{64}$ (b) $\sqrt[3]{-64}$ (c) $\sqrt[5]{-32}$

20. (a) $\sqrt{16}$ (b) $\sqrt[4]{16}$ (c) $\sqrt[4]{\frac{1}{16}}$

21. (a) $\sqrt{\frac{4}{9}}$ (b) $\sqrt[4]{256}$ (c) $\sqrt[6]{\frac{1}{64}}$

22. (a) $\sqrt{7}\sqrt{28}$ (b) $\dfrac{\sqrt{48}}{\sqrt{3}}$ (c) $\sqrt[4]{24}\sqrt[4]{54}$

23. (a) $\left(\frac{4}{9}\right)^{-1/2}$ (b) $(-32)^{2/5}$ (c) $-32^{2/5}$

24. (a) $1024^{-0.1}$ (b) $\left(-\frac{27}{8}\right)^{2/3}$ (c) $\left(\frac{25}{64}\right)^{-3/2}$

25–28 ■ Evaluate the expression using $x = 3$, $y = 4$, and $z = -1$.

25. $\sqrt[4]{x^3 + 14y + 2z}$ **26.** $\sqrt{x^2 + y^2}$

27. $(xy)^{2z}$ **28.** $(9x)^{2/3} + (2y)^{2/3} + z^{2/3}$

29–34 ■ Simplify the expression.

29. $\sqrt{32} + \sqrt{18}$ **30.** $\sqrt{75} + \sqrt{48}$

31. $\sqrt[4]{48} - \sqrt[4]{3}$ **32.** $\sqrt[5]{96} + \sqrt[5]{3}$

33. $\sqrt{16x} + \sqrt{x^5}$ **34.** $\sqrt[3]{2y^4} - \sqrt[3]{y}$

35–40 ■ Simplify each expression.

35. (a) $x^{-5}x^3$ (b) $w^{-2}w^{-4}w^6$ (c) $z^5z^{-3}z^{-4}$

36. (a) x^8x^2 (b) $(3y^2)(4y^5)$ (c) x^2x^{-6}

37. (a) $\dfrac{y^{10}y^0}{y^7}$ (b) $\dfrac{x^6}{x^{10}}$ (c) $\dfrac{a^9a^{-2}}{a}$

38. (a) $\dfrac{z^2z^4}{z^3z^{-1}}$ (b) $(2y^2)^3$ (c) $(8x)^2$

39. (a) $(a^2a^4)^3$ (b) $\left(\frac{a^2}{4}\right)^3$ (c) $(3z)^2(6z^2)^{-3}$

40. (a) $(2z^2)^{-5}z^{10}$ (b) $(2a^3a^2)^4$ (c) $\left(\frac{3x^4}{4x^2}\right)^2$

41–52 ■ Simplify the expression and eliminate any negative exponents(s).

41. (a) $b^4(3ab^3)(2a^2b^{-5})$ (b) $(2s^3t^{-2})(\frac{1}{4}s^7t)(16t^4)$

42. (a) $(4x^2y^4)(2x^5y)$ (b) $(8a^2z)(\frac{1}{2}a^3z^4)$

43. (a) $(5x^2y^3)(3x^2y^5)^4$ (b) $(2a^3b^2)^2(5a^2b^5)^3$

44. (a) $(s^{-2}t^2)^2(s^2t)^3$ (b) $(2u^2v^3)^3(3u^{-3}v)^2$

45. (a) $\frac{2x^3y^4}{x^5y^3}$ (b) $\frac{(2v^3w)^2}{v^3w^2}$

46. (a) $\frac{6y^3z}{2yz^2}$ (b) $\frac{(xy^2z^3)^4}{(x^2y^2z)^3}$

47. (a) $\left(\frac{a^2}{b}\right)^5\left(\frac{a^3b^2}{c^3}\right)^3$ (b) $\frac{(u^{-1}v^2)^2}{(u^3v^{-2})^3}$

48. (a) $\left(\frac{x^4z^2}{4y^5}\right)\left(\frac{2x^3y^2}{z^3}\right)^2$ (b) $\frac{(rs^2)^3}{(r^{-3}s^2)^2}$

49. (a) $\frac{8a^3b^{-4}}{2a^{-5}b^5}$ (b) $\left(\frac{y}{5x^{-2}}\right)^{-3}$

50. (a) $\frac{5xy^{-2}}{x^{-1}y^{-3}}$ (b) $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

51. (a) $\left(\frac{s^2t^{-4}}{5s^{-1}t}\right)^{-2}$ (b) $\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

52. (a) $\left(\frac{3a}{b^3}\right)^{-1}$ (b) $\left(\frac{q^{-1}r^{-1}s^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$

53–60 ■ Simplify the expression. Assume that the letters denote any real numbers.

53. $\sqrt[5]{x^{10}}$ 54. $\sqrt[4]{x^4}$

55. $\sqrt[4]{16x^8}$ 56. $\sqrt[3]{x^3y^6}$

57. $\sqrt[6]{64a^6b^7}$ 58. $\sqrt[3]{a^2b}\sqrt[3]{64a^4b}$

59. $\sqrt[4]{x^4y^2z^2}$ 60. $\sqrt[3]{\sqrt{64x^6}}$

61–70 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

61. (a) $x^{3/4}x^{5/4}$ (b) $y^{2/3}y^{4/3}$

62. (a) $(4b)^{1/2}(8b^{1/4})$ (b) $(3a^{3/4})^2(5a^{1/2})$

63. (a) $\frac{w^{4/3}w^{2/3}}{w^{1/3}}$ (b) $\frac{s^{5/2}(2s^{5/4})^2}{s^{1/2}}$

64. (a) $(8y^3)^{-2/3}$ (b) $(u^4v^6)^{-1/3}$

65. (a) $(x^{-5}y^{1/3})^{-3/5}$ (b) $(2x^3y^{-1/4})^2(8y^{-3/2})^{-1/3}$

66. (a) $(8a^6b^{3/2})^{2/3}$ (b) $(4a^6b^8)^{3/2}$

67. (a) $\frac{(8s^3t^3)^{2/3}}{(s^4t^{-8})^{1/4}}$ (b) $\frac{(32y^{-5}z^{10})^{1/5}}{(64y^6z^{-12})^{-1/6}}$

68. (a) $\left(\frac{x^8y^{-4}}{16y^{4/3}}\right)^{-1/4}$ (b) $\left(\frac{-8y^{3/4}}{y^3z^6}\right)^{-1/3}$

69. (a) $\left(\frac{a^{1/6}b^{-3}}{x^{-1}y}\right)^3\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$ (b) $\frac{(9st)^{3/2}}{(27s^3t^{-4})^{2/3}}\left(\frac{3s^{-2}}{4t^{1/3}}\right)^{-1}$

70. (a) $\left(\frac{x^{-2/3}}{y^{1/2}}\right)\left(\frac{x^{-2}}{y^{-3}}\right)^{1/6}$ (b) $\left(\frac{4y^3z^{2/3}}{x^{1/2}}\right)^2\left(\frac{x^{-3}y^6}{8z^4}\right)^{1/3}$

71–76 ■ Simplify the expression and eliminate any negative exponents(s). Assume that all letters denote positive numbers.

71. (a) $\sqrt[6]{y^5}\sqrt[3]{y^2}$ (b) $(5\sqrt[3]{x})(2\sqrt[4]{x})$

72. (a) $\sqrt[4]{b^3}\sqrt{b}$ (b) $(2\sqrt{a})(\sqrt[3]{a^2})$

73. (a) $\sqrt[5]{x^3y^2}\sqrt[10]{x^4y^{16}}$ (b) $\frac{\sqrt[3]{8x^2}}{\sqrt{x}}$

74. (a) $\sqrt{4st^3}\sqrt[6]{s^3t^2}$ (b) $\frac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$

75. (a) $\sqrt[3]{y\sqrt{y}}$ (b) $\sqrt{\frac{16u^3v}{uv^5}}$

76. (a) $\sqrt{s\sqrt{s^3}}$ (b) $\sqrt[3]{\frac{54x^2y^4}{2x^5y}}$

77–78 ■ Write each number in scientific notation.

77. (a) 129,540,000 (b) 7,259,000,000
(c) 0.0000000014 (d) 0.0007029

78. (a) 69,300,000 (b) 7,200,000,000,000
(c) 0.000028536 (d) 0.0001213

79–80 ■ Write each number in decimal notation.

79. (a) 3.19×10^5 (b) 2.721×10^8
(c) 2.670×10^{-8} (d) 9.999×10^{-9}

80. (a) 7.1×10^{14} (b) 6×10^{12}
(c) 8.55×10^{-3} (d) 6.257×10^{-10}

81–82 ■ Write the number indicated in each statement in scientific notation.

81. (a) A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.
(b) The diameter of an electron is about 0.0000000000004 cm.
(c) A drop of water contains more than 33 billion billion molecules.

82. (a) The distance from the earth to the sun is about 93 million miles.
(b) The mass of an oxygen molecule is about 0.000000000000000000000053 g.
(c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

83–88 ■ Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer rounded to the number of significant digits indicated by the given data.

83. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

84. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

85. $\frac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

86. $\frac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$

87. $\frac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

88. $\frac{(0.0000162)(0.01582)}{(594,621,000)(0.0058)}$

89–92 ■ Rationalize the denominator.

89. (a) $\sqrt{\frac{5}{12}}$ (b) $\sqrt{\frac{x}{6}}$ (c) $\sqrt{\frac{y}{2z}}$

90. (a) $\frac{1}{\sqrt{10}}$ (b) $\sqrt{\frac{2}{x}}$ (c) $\sqrt{\frac{x}{3}}$

91. (a) $\frac{1}{\sqrt[4]{a}}$ (b) $\frac{a}{\sqrt[3]{b^2}}$ (c) $\frac{1}{c^{3/7}}$

92. (a) $\frac{2}{\sqrt[3]{x}}$ (b) $\frac{1}{\sqrt[4]{y^3}}$ (c) $\frac{x}{y^{2/5}}$

93. Let a, b, and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.
(a) b^5 (b) b^{10} (c) ab^2c^3
(d) $(b - a)^3$ (e) $(b - a)^4$ (f) $\frac{a^3c^3}{b^6c^6}$

94. Prove the given Laws of Exponents for the case in which m and n are positive integers and $m > n$.
(a) Law 2 (b) Law 5 (c) Law 6

APPLICATIONS

95. **Speed of Light** The speed of light is about 186,000 mi/s. Use the information in Exercise 82(a) to find how long it takes for a light ray from the sun to reach the earth.

96. **Distance to the Nearest Star** Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 81(a) to express this distance in miles.

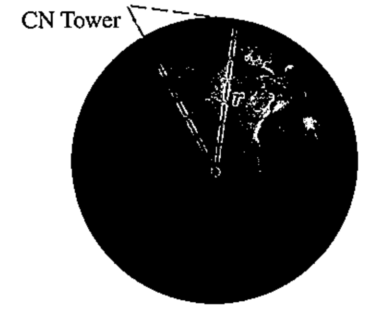
97. **Volume of the Oceans** The average ocean depth is 3.7×10^3 m, and the area of the oceans is 3.6×10^{14} m^2. What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)

98. **National Debt** As of July 2010, the population of the United States was 3.070×10^8, and the national debt was 1.320×10^{13} dollars. How much was each person's share of the debt?

99. **How Far Can You See?** Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. How far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?

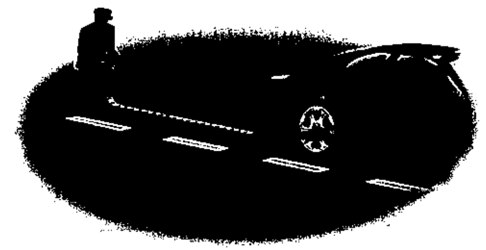


100. **Number of Molecules** A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?

101. **Speed of a Skidding Car** Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, which is a measure of the "slipperiness" of the road. The table gives some typical estimates for f.

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

(a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?
(b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?



102. Distance from the Earth to the Sun It follows from **Kepler's Third Law** of planetary motion that the average distance from a planet to the sun (in meters) is

$$d = \left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$$

where $M = 1.99 \times 10^{30}$ kg is the mass of the sun, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is the gravitational constant, and T is the period of the planet's orbit (in seconds). Use the fact that the period of the earth's orbit is about 365.25 days to find the distance from the earth to the sun.

DISCOVERY ■ DISCUSSION ■ WRITING

103. How Big Is a Billion? If you had a million (10^6) dollars in a suitcase, and you spent a thousand (10^3) dollars each day, how many years would it take you to use all the money? Spending at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

104. Easy Powers That Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\frac{18^5}{9^5}$ (b) $20^6 \cdot (0.5)^6$

105. Limiting Behavior of Powers Complete the following tables. What happens to the nth root of 2 as n gets large? What about the nth root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$\left(\frac{1}{2}\right)^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the nth root of n as n gets large?

106. Comparing Roots Without using a calculator, determine which number is larger in each pair.

(a) $2^{1/2}$ or $2^{1/3}$ (b) $\left(\frac{1}{2}\right)^{1/2}$ or $\left(\frac{1}{2}\right)^{1/3}$

(c) $7^{1/4}$ or $4^{1/3}$ (d) $\sqrt[3]{5}$ or $\sqrt{3}$

1.3 Algebraic Expressions

Adding and Subtracting Polynomials ▶ Multiplying Algebraic Expressions ▶ Special Product Formulas ▶ Factoring Common Factors ▶ Factoring Trinomials ▶ Special Factoring Formulas ▶ Factoring by Grouping Terms

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables, such as x, y, and z and some real numbers, and combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer. A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials. In general, a sum of monomials is called a *polynomial*. For example, the first expression listed above is a polynomial, but the other two are not.

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree** n. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$3 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

▼ Adding and Subtracting Polynomials

We **add** and **subtract** polynomials using the properties of real numbers that were discussed in Section 1.1. The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property. For instance,

$$5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$$

Distributive Property

$ac + bc = (a + b)c$

In subtracting polynomials, we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:

$$-(b + c) = -b - c$$

[This is simply a case of the Distributive Property, $a(b + c) = ab + ac$, with $a = -1$.]

EXAMPLE 1 | Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

SOLUTION

(a) $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$

$= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$ Group like terms

$= 2x^3 - x^2 - 5x + 4$ Combine like terms

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

$= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$ Distributive Property

$= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$ Group like terms

$= -11x^2 + 9x + 4$ Combine like terms

NOW TRY EXERCISES **15** AND **17** ■

▼ Multiplying Algebraic Expressions

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically, we have

$$(a + b)(c + d) = \underset{\mathrm{F}}{ac} + \underset{\mathrm{O}}{ad} + \underset{\mathrm{I}}{bc} + \underset{\mathrm{L}}{bd}$$

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

SOLUTION

(a) $x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4)$ Group terms

$= x^2(x + 1) + 4(x + 1)$ Factor out common factors

$= (x^2 + 4)(x + 1)$ Factor out $x + 1$ from each term

(b) $x^3 - 2x^2 - 3x + 6 = (x^3 - 2x^2) - (3x - 6)$ Group terms

$= x^2(x - 2) - 3(x - 2)$ Factor out common factors

$= (x^2 - 3)(x - 2)$ Factor out $x - 2$ from each term

NOW TRY EXERCISE **83** ■

1.3 EXERCISES

CONCEPTS

1. Consider the polynomial $2x^5 + 6x^4 + 4x^3$.
   How many terms does this polynomial have? ______
   List the terms: ______
   What factor is common to each term? ______
   Factor the polynomial: $2x^5 + 6x^4 + 4x^3 =$ ______.
2. To factor the trinomial $x^2 + 7x + 10$, we look for two integers whose product is ______ and whose sum is ______.
   These integers are ______ and ______, so the trinomial factors as ______.
3. The Special Product Formula for the "square of a sum" is $(A + B)^2 =$ ______.
   So $(2x + 3)^2 =$ ______.
4. The Special Product Formula for the "sum and difference of the same terms" is $(A + B)(A - B) =$ ______.
   So $(5 + x)(5 - x) =$ ______.
5. The Special Factoring Formula for the "difference of squares" is $A^2 - B^2 =$ ______. So $4x^2 - 25$ factors as ______.
6. The Special Factoring Formula for a "perfect square" is $A^2 + 2AB + B^2 =$ ______. So $x^2 + 10x + 25$ factors as ______.

SKILLS

7–12 ■ Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial; then list its terms and state its degree.

Polynomial	Type	Terms	Degree
7. $2x^5 + 4x^2$			
8. $x^2 - 3x + 7$			
9. -8			
10. $\frac{1}{2}x^7$			
11. $\sqrt{2}x - \sqrt{3}$			
12. $x - x^2 + x^3 - x^4$			

13–22 ■ Find the sum, difference, or product.

13. $(5 - 3x) + (2x - 8)$
14. $(12x - 7) - (5x - 12)$
15. $(3x^2 + x + 1) + (2x^2 - 3x - 5)$
16. $(3x^2 + x + 1) - (2x^2 - 3x - 5)$
17. $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$
18. $3(x - 1) + 4(x + 2)$
19. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$
20. $8(2x + 5) - 7(x - 9)$
21. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$
22. $2(2 - 5t) + t^2(t - 1) - (t^4 - 1)$

23–28 ■ Multiply the algebraic expressions using the FOIL method and simplify.

23. $(3t - 2)(7t - 4)$
24. $(4s - 1)(2s + 5)$
25. $(7y - 3)(2y - 1)$
26. $(3x + 5)(2x - 1)$
27. $(4x - 5y)(3x - y)$
28. $(x + 3y)(2x - y)$

29–44 ■ Multiply the algebraic expressions using a Special Product Formula and simplify.

29. $(3x + 4)^2$
30. $(1 - 2y)^2$
31. $(x - 3y)^2$
32. $(2u + v)^2$
33. $(2x + 3y)^2$
34. $(r - 2s)^2$
35. $(y - 3)(y + 3)$
36. $(x + 5)(x - 5)$
37. $(3x - 4)(3x + 4)$
38. $(2y + 5)(2y - 5)$
39. $(\sqrt{y} + \sqrt{2})(\sqrt{y} - \sqrt{2})$
40. $(\sqrt{x} + 2)(\sqrt{x} - 2)$
41. $(y + 2)^3$
42. $(x - 3)^3$
43. $(3 + 2y)^3$
44. $(1 - 2r)^3$

45–60 ■ Perform the indicated operations and simplify.

45. $(x + 2)(x^2 + 2x + 3)$
46. $(x + 1)(2x^2 - x + 1)$
47. $(1 + 2x)(x^2 - 3x + 1)$
48. $(2x - 5)(x^2 - x + 1)$
49. $x^{3/2}(\sqrt{x} - 1/\sqrt{x})$
50. $\sqrt{x}(x - \sqrt{x})$
51. $y^{1/3}(y^{2/3} + y^{5/3})$
52. $x^{1/4}(2x^{3/4} - x^{1/4})$
53. $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$
54. $(x^2 - a^2)(x^2 + a^2)$
55. $(\sqrt{a} - b)(\sqrt{a} + b)$
56. $(\sqrt{h^2 + 1} + 1)(\sqrt{h^2 + 1} - 1)$
57. $(x + (2 + x^2))(x - (2 + x^2))$
58. $((x - 1) + x^2)((x - 1) - x^2)$
59. $(2x + y - 3)(2x + y + 3)$
60. $(x + y + z)(x - y - z)$

61–66 ■ Factor out the common factor.

61. $-2x^3 + 16x$
62. $2x^4 + 4x^3 - 14x^2$
63. $(z + 2)^2 - 5(z + 2)$
64. $y(y - 6) + 9(y - 6)$
65. $2x^2y - 6xy^2 + 3xy$
66. $-7x^4y^2 + 14xy^3 + 21xy^4$

67–74 ■ Factor the trinomial.

67. $x^2 - 6x + 5$
68. $x^2 + 2x - 3$
69. $8x^2 - 14x - 15$
70. $6y^2 + 11y - 21$
71. $3x^2 - 16x + 5$
72. $5x^2 - 7x - 6$
73. $2(a + b)^2 + 5(a + b) - 3$
74. $(3x + 2)^2 + 8(3x + 2) + 12$

75–82 ■ Use a Special Factoring Formula to factor the expression.

75. $9a^2 - 16$
76. $(x + 3)^2 - 4$
77. $a^3 - b^6$
78. $27x^3 + y^3$
79. $8s^3 - 125t^3$
80. $1 + 1000y^3$
81. $16z^2 - 24z + 9$
82. $x^2 + 12x + 36$

83–88 ■ Factor the expression by grouping terms.

83. $x^3 + 4x^2 + x + 4$
84. $3x^3 - x^2 + 6x - 2$
85. $-9x^3 - 3x^2 + 3x + 1$
86. $2x^3 + x^2 - 6x - 3$
87. $x^5 + x^4 + x + 1$
88. $x^3 + x^2 + x + 1$

89–94 ■ Factor the expression completely. Begin by factoring out the lowest power of each common factor.

89. $3x^{-1/2} + 4x^{1/2} + x^{3/2}$
90. $x^{5/2} - x^{1/2}$
91. $x^{-3/2} + 2x^{-1/2} + x^{1/2}$
92. $(x - 1)^{7/2} - (x - 1)^{3/2}$
93. $x^{-1/2}(x + 1)^{1/2} + x^{1/2}(x + 1)^{-1/2}$
94. $(x^2 + 1)^{1/2} + 2(x^2 + 1)^{-1/2}$

95–124 ■ Factor the expression completely.

95. $30x^3 + 15x^4$
96. $12x^3 + 18x$
97. $x^2 - 14x + 48$
98. $x^2 - 2x - 8$
99. $2x^2 + 7x - 4$
100. $2x^2 + 5x + 3$
101. $9x^2 - 36x - 45$
102. $8x^2 + 10x + 3$
103. $4t^2 - 9s^2$
104. $49 - 4y^2$
105. $t^2 - 6t + 9$
106. $x^2 + 10x + 25$
107. $4x^2 + 4xy + y^2$
108. $r^2 - 6rs + 9s^2$
109. $(a + b)^2 - (a - b)^2$
110. $\left(1 + \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)^2$
111. $(a^2 - 1)b^2 - 4(a^2 - 1)$
112. $x^2(x^2 - 1) - 9(x^2 - 1)$
113. $x^6 + 64$
114. $8x^3 - 125$
115. $x^3 + 2x^2 + x$
116. $3x^3 - 27x$
117. $x^4y^3 - x^2y^5$
118. $18y^3x^2 - 2xy^4$
119. $3x^3 + 5x^2 - 6x - 10$
120. $2x^3 + 4x^2 + x + 2$
121. $y^4(y + 2)^3 + y^5(y + 2)^4$
122. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$
123. $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$
124. $(a^2 + 1)^2 - 7(a^2 + 1) + 10$

125–128 ■ Factor the expression completely. (This type of expression arises in calculus when using the "Product Rule.")

125. $3(2x - 1)^2(2)(x + 3)^{1/2} + (2x - 1)^3(\frac{1}{2})(x + 3)^{-1/2}$
126. $5(x^2 + 4)^4(2x)(x - 2)^4 + (x^2 + 4)^5(4)(x - 2)^3$
127. $\frac{1}{2}x^{-1/2}(3x + 4)^{1/2} - \frac{3}{2}x^{1/2}(3x + 4)^{-1/2}$
128. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$
129. (a) Show that $ab = \frac{1}{2}[(a + b)^2 - (a^2 + b^2)]$.
     (b) Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2$.
     (c) Show that
     $$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$
     (d) Factor completely: $4a^2c^2 - (a^2 - b^2 + c^2)^2$.
130. Verify Special Factoring Formulas 4 and 5 by expanding their right-hand sides.

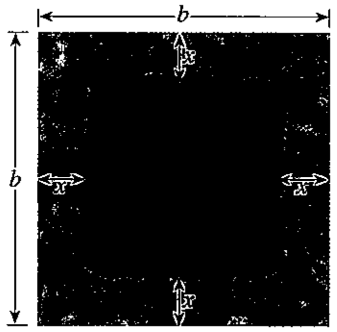
APPLICATIONS

131. **Volume of Concrete** A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside front cover of this book, explain why the volume of the cylindrical shell is
$$V = \pi R^2h - \pi r^2h$$
Factor to show that
$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$
Use the "unrolled" diagram to explain why this makes sense geometrically.

132. Mowing a Field A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as a habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

(a) Explain why the area of the mowed portion is $b^2 - (b - 2x)^2$.

(b) Factor the expression in part (a) to show that the area of the mowed portion is also $4x(b - x)$.



DISCOVERY ▪ DISCUSSION ▪ WRITING

133. Degrees of Sums and Products of Polynomials Make up several pairs of polynomials, then calculate the sum and product of each pair. On the basis of your experiments and observations, answer the following questions.

(a) How is the degree of the product related to the degrees of the original polynomials?

(b) How is the degree of the sum related to the degrees of the original polynomials?

134. The Power of Algebraic Formulas Use the Difference of Squares Formula to factor $17^2 - 16^2$. Notice that it is easy to calculate the factored form in your head but not so easy to calculate the original form in this way. Evaluate each expression in your head:

(a) $528^2 - 527^2$

(b) $122^2 - 120^2$

(c) $1020^2 - 1010^2$

Now use the Special Product Formula

$$(A + B)(A - B) = A^2 - B^2$$

to evaluate these products in your head:

(d) $79 \cdot 51$

(e) $998 \cdot 1002$

135. Differences of Even Powers

(a) Factor the expressions completely: $A^4 - B^4$ and $A^6 - B^6$.

(b) Verify that $18{,}335 = 12^4 - 7^4$ and that $2{,}868{,}335 = 12^6 - 7^6$.

(c) Use the results of parts (a) and (b) to factor the integers 18,335 and 2,868,335. Then show that in both of these factorizations, all the factors are prime numbers.

136. Factoring $A^n - 1$ Verify these formulas by expanding and simplifying the right-hand side.

$$A^2 - 1 = (A - 1)(A + 1)$$
$$A^3 - 1 = (A - 1)(A^2 + A + 1)$$
$$A^4 - 1 = (A - 1)(A^3 + A^2 + A + 1)$$

On the basis of the pattern displayed in this list, how do you think $A^5 - 1$ would factor? Verify your conjecture. Now generalize the pattern you have observed to obtain a factoring formula for $A^n - 1$, where n is a positive integer.

137. Factoring $x^4 + ax^2 + b$ A trinomial of the form $x^4 + ax^2 + b$ can sometimes be factored easily. For example,

$$x^4 + 3x^2 - 4 = (x^2 + 4)(x^2 - 1)$$

But $x^4 + 3x^2 + 4$ cannot be factored in this way. Instead, we can use the following method.

$$x^4 + 3x^2 + 4 = (x^4 + 4x^2 + 4) - x^2 \quad \text{Add and subtract } x^2$$
$$= (x^2 + 2)^2 - x^2 \quad \text{Factor perfect square}$$
$$= [(x^2 + 2) - x][(x^2 + 2) + x] \quad \text{Difference of squares}$$
$$= (x^2 - x + 2)(x^2 + x + 2)$$

Factor the following, using whichever method is appropriate.

(a) $x^4 + x^2 - 2$

(b) $x^4 + 2x^2 + 9$

(c) $x^4 + 4x^2 + 16$

(d) $x^4 + 2x^2 + 1$

DISCOVERY PROJECT **Visualizing a Formula**

In this project we discover geometric interpretations of some of the Special Product Formulas. You can find the project at the book companion website: **www.cengage.com/international**

1.4 RATIONAL EXPRESSIONS

The Domain of an Algebraic Expression ▶ Simplifying Rational Expressions ▶ Multiplying and Dividing Rational Expressions ▶ Adding and Subtracting Rational Expressions ▶ Compound Fractions ▶ Rationalizing the Denominator or the Numerator ▶ Avoiding Common Errors

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \qquad \frac{\sqrt{x}+3}{x+1} \qquad \frac{y-2}{y^2+4}$$

A **rational expression** is a fractional expression where both the numerator and denominator are polynomials. For example, the following are rational expressions:

$$\frac{2x}{x-1} \qquad \frac{x}{x^2+1} \qquad \frac{x^3-x}{x^2-5x+6}$$

In this section we learn how to perform algebraic operations on rational expressions.

▼ The Domain of an Algebraic Expression

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

EXAMPLE 1 | Finding the Domain of an Expression

Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ (b) $\dfrac{x}{x^2 - 5x + 6}$ (c) $\dfrac{\sqrt{x}}{x-5}$

SOLUTION

(a) This polynomial is defined for every x. Thus, the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator.

$$\frac{x}{x^2 - 5x + 6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root

$$\frac{\sqrt{x}}{x-5}$$

Denominator would be 0 if $x = 5$

Thus, the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

NOW TRY EXERCISE 11 ■

SOLUTION 2 Since $(1 + x^2)^{-1/2} = 1/(1 + x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1 + x^2)^{1/2}$.

$$\frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} = \frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} \cdot \frac{(1 + x^2)^{1/2}}{(1 + x^2)^{1/2}}$$

$$= \frac{(1 + x^2) - x^2}{(1 + x^2)^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

NOW TRY EXERCISE 77 ■

▼ Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A + B\sqrt{C}$, we may rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This works because, by Special Product Formula 1 in Section 1.3, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

EXAMPLE 9 | Rationalizing the Denominator

Rationalize the denominator: $\dfrac{1}{1 + \sqrt{2}}$

SOLUTION We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$\frac{1}{1 + \sqrt{2}} = \frac{1}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}} \qquad \text{Multiply numerator and denominator by the conjugate radical}$$

$$= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2} \qquad \text{Special Product Formula 1}$$

$$= \frac{1 - \sqrt{2}}{1 - 2} = \frac{1 - \sqrt{2}}{-1} = \sqrt{2} - 1$$

NOW TRY EXERCISE 81 ■

EXAMPLE 10 | Rationalizing the Numerator

Rationalize the numerator: $\dfrac{\sqrt{4 + h} - 2}{h}$

SOLUTION We multiply numerator and denominator by the conjugate radical $\sqrt{4 + h} + 2$.

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$\frac{\sqrt{4 + h} - 2}{h} = \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} \qquad \text{Multiply numerator and denominator by the conjugate radical}$$

$$= \frac{(\sqrt{4 + h})^2 - 2^2}{h(\sqrt{4 + h} + 2)} \qquad \text{Special Product Formula 1}$$

$$= \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)}$$

$$= \frac{h}{h(\sqrt{4 + h} + 2)} = \frac{1}{\sqrt{4 + h} + 2} \qquad \text{Property 5 of fractions (cancel common factors)}$$

NOW TRY EXERCISE 87 ■

▼ Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \neq a + b$
$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$	$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a + b}$
$\frac{ab}{a} = b$	$\frac{a + b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \neq (a + b)^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we find that the left-hand side is

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1$$

whereas the right-hand side is

$$\frac{1}{a + b} = \frac{1}{2 + 2} = \frac{1}{4}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercise 105.)

1.4 EXERCISES

CONCEPTS

1. Which of the following are rational expressions?

(a) $\dfrac{3x}{x^2 - 1}$ (b) $\dfrac{\sqrt{x + 1}}{2x + 3}$ (c) $\dfrac{x(x^2 - 1)}{x + 3}$

2. To simplify a rational expression, we cancel *factors* that are common to the _______ and _______. So the expression

$$\frac{(x + 1)(x + 2)}{(x + 3)(x + 2)}$$

simplifies to _______.

3. To multiply two rational expressions, we multiply their _______ together and multiply their _______ together. So $\dfrac{2}{x + 1} \cdot \dfrac{x}{x + 3}$ is the same as _______.

4. Consider the expression $\dfrac{1}{x} - \dfrac{2}{x + 1} - \dfrac{x}{(x + 1)^2}$.

(a) How many terms does this expression have?
(b) Find the least common denominator of all the terms.
(c) Perform the addition and simplify.

SKILLS

5–12 ■ Find the domain of the expression.

5. $-x^4 + x^3 + 9x$
6. $4x^2 - 10x + 3$
7. $\dfrac{2x + 1}{x - 4}$
8. $\dfrac{2t^2 - 5}{3t + 6}$
9. $\dfrac{1}{\sqrt{x - 1}}$
10. $\sqrt{x + 3}$
11. $\dfrac{x^2 + 1}{x^2 - x - 2}$
12. $\dfrac{\sqrt{2x}}{x + 1}$

13–22 ■ Simplify the rational expression.

13. $\dfrac{4(x^2-1)}{12(x+2)(x-1)}$

14. $\dfrac{3(x+2)(x-1)}{6(x-1)^2}$

15. $\dfrac{x^2-x-2}{x^2-1}$

16. $\dfrac{x-2}{x^2-4}$

17. $\dfrac{x^2+6x+8}{x^2+5x+4}$

18. $\dfrac{x^2-x-12}{x^2+5x+6}$

19. $\dfrac{y^2-3y-18}{2y^2+5y+3}$

20. $\dfrac{y^2+y}{y^2-1}$

21. $\dfrac{2x^3-x^2-6x}{2x^2-7x+6}$

22. $\dfrac{1-x^2}{x^3-1}$

23–38 ■ Perform the multiplication or division and simplify.

23. $\dfrac{x^2-25}{x^2-16}\cdot\dfrac{x+4}{x+5}$

24. $\dfrac{4x}{x^2-4}\cdot\dfrac{x+2}{16x}$

25. $\dfrac{x^2-2x-15}{x^2-9}\cdot\dfrac{x+3}{x-5}$

26. $\dfrac{x^2+2x-3}{x^2-2x-3}\cdot\dfrac{3-x}{3+x}$

27. $\dfrac{x^2-x-6}{x^2+2x}\cdot\dfrac{x^3+x^2}{x^2-2x-3}$

28. $\dfrac{t-3}{t^2+9}\cdot\dfrac{t+3}{t^2-9}$

29. $\dfrac{x^2+2xy+y^2}{x^2-y^2}\cdot\dfrac{2x^2-xy-y^2}{x^2-xy-2y^2}$

30. $\dfrac{x^2+7x+12}{x^2+3x+2}\cdot\dfrac{x^2+5x+6}{x^2+6x+9}$

31. $\dfrac{x+3}{4x^2-9}\div\dfrac{x^2+7x+12}{2x^2+7x-15}$

32. $\dfrac{2x+1}{2x^2+x-15}\div\dfrac{6x^2-x-2}{x+3}$

33. $\dfrac{4y^2-9}{2y^2+9y-18}\div\dfrac{2y^2+y-3}{y^2+5y-6}$

34. $\dfrac{2x^2+3x+1}{x^2+2x-15}\div\dfrac{x^2+6x+5}{2x^2-7x+3}$

35. $\dfrac{\dfrac{x^3}{x+1}}{\dfrac{x}{x^2+2x+1}}$

36. $\dfrac{\dfrac{2x^2-3x-2}{x^2-1}}{\dfrac{2x^2+5x+2}{x^2+x-2}}$

37. $\dfrac{x}{y/z}$

38. $\dfrac{x/y}{z}$

39–58 ■ Perform the addition or subtraction and simplify.

39. $2+\dfrac{x}{x+3}$

40. $\dfrac{2x-1}{x+4}-1$

41. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$

42. $\dfrac{1}{x+5}+\dfrac{2}{x-3}$

43. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$

44. $\dfrac{x}{x-4}-\dfrac{3}{x+6}$

45. $\dfrac{x}{(x+1)^2}+\dfrac{2}{x+1}$

46. $\dfrac{5}{2x-3}-\dfrac{3}{(2x-3)^2}$

47. $\dfrac{2}{a^2}-\dfrac{3}{ab}+\dfrac{4}{b^2}$

48. $u+1+\dfrac{u}{u+1}$

49. $\dfrac{1}{x^2}+\dfrac{1}{x^2+x}$

50. $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}$

51. $\dfrac{x}{x^2-4}+\dfrac{1}{x-2}$

52. $\dfrac{2}{x+3}-\dfrac{1}{x^2+7x+12}$

53. $\dfrac{x}{x^2+x-2}-\dfrac{2}{x^2-5x+4}$

54. $\dfrac{1}{x+3}+\dfrac{1}{x^2-9}$

55. $\dfrac{2}{x}+\dfrac{3}{x-1}-\dfrac{4}{x^2-x}$

56. $\dfrac{x}{x^2-x-6}-\dfrac{1}{x+2}-\dfrac{2}{x-3}$

57. $\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}+\dfrac{3}{x^2-1}$

58. $\dfrac{1}{x^2+3x+2}-\dfrac{1}{x^2-2x-3}$

59–68 ■ Simplify the compound fractional expression.

59. $\dfrac{x+\dfrac{1}{x+2}}{x-\dfrac{1}{x+2}}$

60. $\dfrac{1+\dfrac{1}{c-1}}{1-\dfrac{1}{c-1}}$

61. $\dfrac{\dfrac{x+2}{x-1}-\dfrac{x-3}{x-2}}{x+2}$

62. $\dfrac{\dfrac{x-3}{x-4}-\dfrac{x+2}{x+1}}{x+3}$

63. $x-\dfrac{y}{\dfrac{x}{y}+\dfrac{y}{x}}$

64. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$

65. $\dfrac{x^{-1}+y^{-1}}{(x+y)^{-1}}$

66. $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$

67. $1-\dfrac{1}{1-\dfrac{1}{x}}$

68. $1+\dfrac{1}{1+\dfrac{1}{1+x}}$

69–74 ■ Simplify the fractional expression. (Expressions like these arise in calculus.)

69. $\dfrac{\dfrac{1}{1+x+h}-\dfrac{1}{1+x}}{h}$

70. $\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

71. $\dfrac{(x+h)^3-7(x+h)-(x^3-7x)}{h}$

72. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

73. $\sqrt{1+\left(x^3-\dfrac{1}{4x^3}\right)^2}$

74. $\sqrt{1+\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}$

75–80 ■ Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

75. $\dfrac{2x(x+6)^4-x^2(4)(x+6)^3}{(x+6)^8}$

76. $\dfrac{3(x+2)^2(x-3)^2-(x+2)^3(2)(x-3)}{(x-3)^4}$

77. $\dfrac{2(1+x)^{1/2}-x(1+x)^{-1/2}}{x+1}$

78. $\dfrac{(1-x^2)^{1/2}+x^2(1-x^2)^{-1/2}}{1-x^2}$

79. $\dfrac{(7-3x)^{1/2}+\frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$

80. $\dfrac{3(1+x)^{1/3}-x(1+x)^{-2/3}}{(1+x)^{2/3}}$

81–86 ■ Rationalize the denominator.

81. $\dfrac{1}{2-\sqrt{3}}$

82. $\dfrac{2}{3-\sqrt{5}}$

83. $\dfrac{1}{\sqrt{x}+1}$

84. $\dfrac{2}{\sqrt{2}+\sqrt{7}}$

85. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$

86. $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

87–92 ■ Rationalize the numerator.

87. $\dfrac{1-\sqrt{5}}{3}$

88. $\dfrac{\sqrt{3}+\sqrt{5}}{2}$

89. $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

90. $\dfrac{\sqrt{r}+\sqrt{2}}{5}$

91. $\sqrt{x^2+1}-x$

92. $\sqrt{x+1}-\sqrt{x}$

93–100 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator zero.)

93. $\dfrac{b}{b-c}=1-\dfrac{b}{c}$

94. $\dfrac{16+a}{16}=1+\dfrac{a}{16}$

95. $\dfrac{2}{4+x}=\dfrac{1}{2}+\dfrac{2}{x}$

96. $\dfrac{x+1}{y+1}=\dfrac{x}{y}$

97. $2\left(\dfrac{a}{b}\right)=\dfrac{2a}{2b}$

98. $\dfrac{x}{x+y}=\dfrac{1}{1+y}$

99. $\dfrac{-a}{b}=-\dfrac{a}{b}$

100. $\dfrac{1+x+x^2}{x}=\dfrac{1}{x}+1+x$

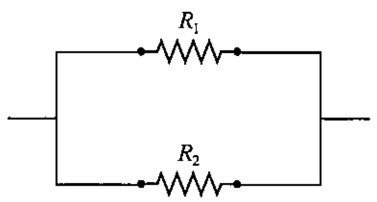
APPLICATIONS

101. **Electrical Resistance** If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R=\dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$$

(a) Simplify the expression for R.

(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?



102. **Average Cost** A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.

(a) Explain why the average cost per shirt is given by the rational expression

$$A=\dfrac{500+6x+0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

DISCOVERY ■ DISCUSSION ■ WRITING

103. **Limiting Behavior of a Rational Expression** The rational expression

$$\dfrac{x^2-9}{x-3}$$

is not defined for $x = 3$. Complete the tables and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\dfrac{x^2-9}{x-3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\dfrac{x^2-9}{x-3}$
3.20	
3.10	
3.05	
3.01	
3.001	

104. **Is This Rationalization?** In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator?

105. **Algebraic Errors** The left-hand column in the table on the following page lists some common algebraic errors. In each case, give an example using numbers that show that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic error	Counterexample
$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$	$\frac{1}{2} + \frac{1}{2} \neq \frac{1}{2+2}$
$(a+b)^2 \neq a^2 + b^2$	
$\sqrt{a^2+b^2} \neq a + b$	
$\frac{a+b}{a} \neq b$	
$(a^3+b^3)^{1/3} \neq a + b$	
$a^m/a^n \neq a^{m/n}$	
$a^{-1/n} \neq \frac{1}{a^n}$	

106. The Form of an Algebraic Expression An algebraic expression may look complicated, but its "form" is always simple; it must be a sum, a product, a quotient, or a power. For example, consider the following expressions:

$$(1+x^2)^2 + \left(\frac{x+2}{x+1}\right)^3 \qquad (1+x)\left(1+\frac{x+5}{1+x^4}\right)$$

$$\frac{5-x^3}{1+\sqrt{1+x^2}} \qquad \sqrt{\frac{1+x}{1-x}}$$

With appropriate choices for A and B, the first has the form $A + B$, the second AB, the third A/B, and the fourth $A^{1/2}$. Recognizing the form of an expression helps us expand, simplify, or factor it correctly. Find the form of the following algebraic expressions.

(a) $x + \sqrt{1 + \frac{1}{x}}$ (b) $(1 + x^2)(1 + x)^3$

(c) $\sqrt[3]{x^4(4x^2+1)}$ (d) $\dfrac{1 - 2\sqrt{1+x}}{1+\sqrt{1+x^2}}$

1.5 Equations

Solving Linear Equations ▶ Solving Quadratic Equations ▶ Other Types of Equations

An equation is a statement that two mathematical expressions are equal. For example,

$$3 + 5 = 8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x + 7 = 19$$

the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

$x = 3$ is a solution of the equation $4x + 7 = 19$, because substituting $x = 3$ makes the equation true:

$x = 3$

$4(3) + 7 = 19$ ✔

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the "equal" sign. Here are the properties that we use to solve an equation. (In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "is equivalent to.")

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \Leftrightarrow A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \Leftrightarrow CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus, if we say "*add* -7" when solving an equation, that is just a short way of saying "*add* -7 to each side of the equation."

▼ Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \frac{x}{3}$	$\frac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

EXAMPLE 1 | Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

SOLUTION We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$7x - 4 = 3x + 8$	Given equation
$(7x - 4) + 4 = (3x + 8) + 4$	Add 4
$7x = 3x + 12$	Simplify
$7x - 3x = (3x + 12) - 3x$	Subtract $3x$
$4x = 12$	Simplify
$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 12$	Multiply by $\frac{1}{4}$
$x = 3$	Simplify

Because it is important to CHECK YOUR ANSWER, we do this in many of our examples. In these checks, LHS stands for "left-hand side" and RHS stands for "right-hand side" of the original equation.

CHECK YOUR ANSWER

$x = 3$:

	$x = 3$	$x = 3$
	LHS $= 7(3) - 4$	RHS $= 3(3) + 8$
	$= 17$	$= 17$

LHS = RHS ✔

NOW TRY EXERCISE 15 ■

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In the next example we solve for a variable in Newton's Law of Gravity.

When solving equations that involve absolute values, we usually take cases.

EXAMPLE 14 | An Absolute Value Equation

Solve the equation $|2x - 5| = 3$.

SOLUTION By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$
$$2x = 8 \qquad\qquad 2x = 2$$
$$x = 4 \qquad\qquad x = 1$$

The solutions are $x = 1$, $x = 4$.

NOW TRY EXERCISE 105 ■

1.5 EXERCISES

CONCEPTS

1. *True or false?*
   (a) Adding the same number to each side of an equation always gives an equivalent equation.
   (b) Multiplying each side of an equation by the same number always gives an equivalent equation.
   (c) Squaring each side of an equation always gives an equivalent equation.

2. Explain how you would use each method to solve the equation $x^2 - 4x - 5 = 0$.
   (a) By factoring: ________
   (b) By completing the square: ________
   (c) By using the Quadratic Formula: ________

3. (a) The solutions of the equation $x^2(x - 4) = 0$ are ________.
   (b) To solve the equation $x^3 - 4x^2 = 0$, we ________ the left-hand side.

4. Solve the equation $\sqrt{2x} + x = 0$ by doing the following steps.
   (a) Isolate the radical: ________________.
   (b) Square both sides: ________________.
   (c) The solutions of the resulting quadratic equation are ________.
   (d) The solution(s) that satisfy the original equation are ________.

5. The equation $(x + 1)^2 - 5(x + 1) + 6 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________.

6. The equation $x^6 + 7x^3 - 8 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________.

SKILLS

7–10 ■ Determine whether the given value is a solution of the equation.

7. $4x + 7 = 9x - 3$
   (a) $x = -2$ (b) $x = 2$
8. $1 - [2 - (3 - x)] = 4x - (6 + x)$
   (a) $x = 2$ (b) $x = 4$
9. $\dfrac{x^{3/2}}{x - 6} = x - 8$
   (a) $x = 4$ (b) $x = 8$
10. $\dfrac{1}{x} - \dfrac{1}{x - 4} = 1$
   (a) $x = 2$ (b) $x = 4$

11–28 ■ The given equation is either linear or equivalent to a linear equation. Solve the equation.

11. $5x - 3 = 4$
12. $2x + 7 = 31$
13. $3 + \frac{1}{3}x = 5$
14. $\frac{1}{2}x - 8 = 1$
15. $-7w = 15 - 2w$
16. $5t - 13 = 12 - 5t$
17. $\frac{1}{2}y - 2 = \frac{1}{3}y$
18. $\dfrac{z}{5} = \dfrac{3}{10}z + 7$
19. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$
20. $2(1 - x) = 3(1 + 2x) + 5$
21. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$
22. $2x - \dfrac{x}{2} + \dfrac{x + 1}{4} = 6x$
23. $\dfrac{2x - 1}{x + 2} = \dfrac{4}{5}$
24. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$
25. $\dfrac{4}{x - 1} + \dfrac{2}{x + 1} = \dfrac{35}{x^2 - 1}$
26. $\dfrac{3}{x + 1} - \dfrac{1}{2} = \dfrac{1}{3x + 3}$
27. $(t - 4)^2 = (t + 4)^2 + 32$
28. $\sqrt{3}x + \sqrt{12} = \dfrac{x + 5}{\sqrt{3}}$

29–42 ■ Solve the equation for the indicated variable.

29. $PV = nRT$; for R
30. $F = G\dfrac{mM}{r^2}$; for m

31. $P = 2l + 2w$; for w
32. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1
33. $a - 2[b - 3(c - x)] = 6$; for x
34. $\dfrac{ax + b}{cx + d} = 2$; for x
35. $a^2x + (a - 1) = (a + 1)x$; for x
36. $\dfrac{a + 1}{b} = \dfrac{a - 1}{b} + \dfrac{b + 1}{a}$; for a
37. $V = \frac{1}{3}\pi r^2 h$; for r
38. $F = G\dfrac{mM}{r^2}$; for r
39. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i
40. $a^2 + b^2 = c^2$; for b
41. $h = \frac{1}{2}gt^2 + v_0t$; for t
42. $S = \dfrac{n(n + 1)}{2}$; for n

43–54 ■ Solve the equation by factoring.

43. $x^2 + x - 12 = 0$
44. $x^2 + 3x - 4 = 0$
45. $x^2 + 8x + 12 = 0$
46. $x^2 - 7x + 12 = 0$
47. $2y^2 + 7y + 3 = 0$
48. $4x^2 - 4x - 15 = 0$
49. $3x^2 + 5x = 2$
50. $6x(x - 1) = 21 - x$
51. $3x^2 - 27 = 0$
52. $2x^2 = 8$
53. $(2x - 1)^2 = 8$
54. $(3x + 2)^2 = 10$

55–62 ■ Solve the equation by completing the square.

55. $x^2 + 2x - 5 = 0$
56. $x^2 - 4x + 2 = 0$
57. $x^2 + 3x - \frac{7}{4} = 0$
58. $x^2 - 6x - 11 = 0$
59. $2x^2 + 8x + 1 = 0$
60. $3x^2 - 6x - 1 = 0$
61. $x^2 = \frac{3}{4}x - \frac{1}{8}$
62. $4x^2 - x = 0$

63–78 ■ Find all real solutions of the quadratic equation.

63. $x^2 + 5x - 6 = 0$
64. $x^2 - 2x - 15 = 0$
65. $x^2 - 7x + 10 = 0$
66. $x^2 + 30x + 200 = 0$
67. $3x^2 + 7x + 4 = 0$
68. $2x^2 + x - 3 = 0$
69. $3x^2 + 6x - 5 = 0$
70. $x^2 - 6x + 1 = 0$
71. $2y^2 - y - \frac{1}{2} = 0$
72. $z^2 - \frac{3}{2}z + \frac{9}{16} = 0$
73. $0 = x^2 - 4x + 1$
74. $4x^2 + 16x - 9 = 0$
75. $w^2 = 3(w - 1)$
76. $3 + 5z + z^2 = 0$
77. $25x^2 + 70x + 49 = 0$
78. $10y^2 - 16y + 5 = 0$

79–84 ■ Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

79. $x^2 - 6x + 1 = 0$
80. $3x^2 = 6x - 9$
81. $x^2 + 2.20x + 1.21 = 0$
82. $x^2 + 2.21x + 1.21 = 0$
83. $4x^2 + 5x + \frac{13}{8} = 0$
84. $x^2 + rx - s = 0 \quad (s > 0)$

85–108 ■ Find all real solutions of the equation.

85. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} = \dfrac{5}{4}$
86. $\dfrac{10}{x} - \dfrac{12}{x - 3} + 4 = 0$
87. $\dfrac{1}{x - 1} - \dfrac{2}{x^2} = 0$
88. $\dfrac{x^2}{x + 100} = 50$
89. $\dfrac{x}{2x + 7} - \dfrac{x + 1}{x + 3} = 1$
90. $\dfrac{x + 5}{x - 2} = \dfrac{5}{x + 2} + \dfrac{28}{x^2 - 4}$
91. $\sqrt{2x + 1} + 1 = x$
92. $\sqrt{5 - x} + 1 = x - 2$
93. $\sqrt{\sqrt{x - 5} + x} = 5$
94. $2x + \sqrt{x + 1} = 8$
95. $x^4 - 13x^2 + 40 = 0$
96. $x^4 - 5x^2 + 4 = 0$
97. $2x^4 + 4x^2 + 1 = 0$
98. $x^6 - 2x^3 - 3 = 0$
99. $x^{4/3} - 5x^{2/3} + 6 = 0$
100. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$
101. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$
102. $4(x + 1)^{1/2} - 5(x + 1)^{3/2} + (x + 1)^{5/2} = 0$
103. $x - 5\sqrt{x} + 6 = 0$
104. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$
105. $|3x + 5| = 1$
106. $|2x| = 3$
107. $|x - 6| = -1$
108. $|x - 4| = 0.01$

APPLICATIONS

109–110 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

109. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

110. A ball is dropped from the top of a building 96 ft tall.
   (a) How long will it take to fall half the distance to ground level?
   (b) How long will it take to fall to ground level?

111–112 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0t$ discussed in Example 9.

111. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.
   (a) When does the ball reach a height of 24 ft?
   (b) When does it reach a height of 48 ft?
   (c) What is the greatest height reached by the ball?
   (d) When does the ball reach the highest point of its path?
   (e) When does the ball hit the ground?

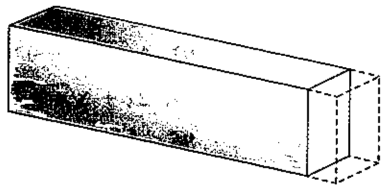
112. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0t + h = 0$.]

113. **Shrinkage in Concrete Beams** As concrete dries, it shrinks—the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m³, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.
   (a) A beam 12.025 m long is cast in concrete that contains 250 kg/m³ water. What is the shrinkage factor S? How long will the beam be when it has dried?

(b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



114. **The Lens Equation** If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

115. **Fish Population** A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?

116. **Fish Population** The fish population in a certain lake rises and falls according to the formula

$$F = 1000(30 + 17t - t^2)$$

Here F is the number of fish at time t, where t is measured in years since January 1, 2002, when the fish population was first estimated.

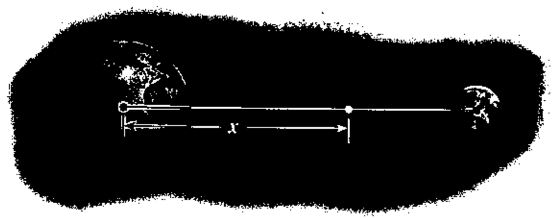
(a) On what date will the fish population again be the same as it was on January 1, 2002?

(b) By what date will all the fish in the lake have died?

117. **Gravity** If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2}$$

where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the "dead spot" where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



118. **Profit** A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$ provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of \$1250?

119. **Depth of a Well** One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes for the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes for the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus, the total time elapsed between dropping the stone and hearing the splash is

$$t_1 + t_2 = \frac{\sqrt{d}}{4} + \frac{d}{1090}$$

How deep is the well if this total time is 3 s?

DISCOVERY ■ DISCUSSION ■ WRITING

120. **A Family of Equations** The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k, we get a different equation with the unknown x. The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

(a) $x = 0$ (b) $x = 1$ (c) $x = 2$

121. **Proof That 0 = 1?** The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

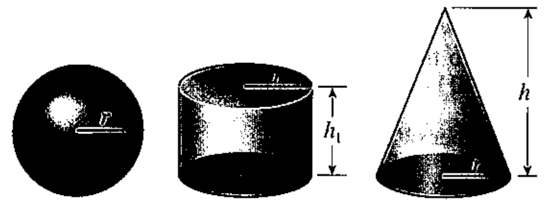
$x = 1$	Given
$x^2 = x$	Multiply by x
$x^2 - x = 0$	Subtract x
$x(x - 1) = 0$	Factor
$\frac{x(x-1)}{x-1} = \frac{0}{x-1}$	Divide by $x - 1$
$x = 0$	Simplify
$1 = 0$	Given $x = 1$

122. **Volumes of Solids** The sphere, cylinder, and cone shown here all have the same radius r and the same volume V.

(a) Use the volume formulas given on the inside front cover of this book, to show that

$$\tfrac{4}{3}\pi r^3 = \pi r^2 h_1 \quad \text{and} \quad \tfrac{4}{3}\pi r^3 = \tfrac{1}{3}\pi r^2 h_2$$

(b) Solve these equations for h_1 and h_2.



123. **Relationship Between Roots and Coefficients** The Quadratic Formula gives us the roots of a quadratic equation from its coefficients. We can also obtain the coefficients from the roots. For example, find the roots of the equation $x^2 - 9x + 20 = 0$ and show that the product of the roots is the constant term 20 and the sum of the roots is 9, the negative of the coefficient of x. Show that the same relationship between roots and coefficients holds for the following equations:

$$x^2 - 2x - 8 = 0$$
$$x^2 + 4x + 2 = 0$$

Use the Quadratic Formula to prove that in general, if the equation $x^2 + bx + c = 0$ has roots r_1 and r_2, then $c = r_1r_2$ and $b = -(r_1 + r_2)$.

124. **Solving an Equation in Different Ways** We have learned several different ways to solve an equation in this section. Some equations can be tackled by more than one method. For example, the equation $x - \sqrt{x} - 2 = 0$ is of quadratic type. We can solve it by letting $\sqrt{x} = u$ and $x = u^2$, and factoring. Or we could solve for $\sqrt{x}$, square each side, and then solve the resulting quadratic equation. Solve the following equations using both methods indicated, and show that you get the same final answers.

(a) $x - \sqrt{x} - 2 = 0$ quadratic type; solve for the radical, and square

(b) $\frac{12}{(x-3)^2} + \frac{10}{x-3} + 1 = 0$ quadratic type; multiply by LCD

1.6 Modeling with Equations

Making and Using Models ▶ Problems About Interest ▶ Problems About Area or Length ▶ Problems About Mixtures ▶ Problems About the Time Needed to Do a Job ▶ Problems About Distance, Rate, and Time

Many problems in the sciences, economics, finance, medicine, and numerous other fields can be translated into algebra problems; this is one reason that algebra is so useful. In this section we use equations as mathematical models to solve real-life problems.

▼ Making and Using Models

We will use the following guidelines to help us set up equations that model situations described in words. To show how the guidelines can help you to set up equations, we note them as we work each example in this section.

GUIDELINES FOR MODELING WITH EQUATIONS

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question that is posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
2. **Translate from Words to Algebra.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table.**
3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.