12.1 (P.792) 5th — (CW: turn in at end of class)

11.1 (Pg. 830) # 5, 8, 11, 14, 15, 16, 21, 25, 27, 33, 37, 38, 41, 43, 45, 49, 55, 56, 58, 62, 63, 64, 65, 67, 73.

Seq, Σ 6th edition: 7, 10, 13, 15, 17, 18, 24, 27, 29, 36, 39, 40, 43, 46, 48, 52, 58, 57, 59, 64, 66, 65, 67, 69, 76

arithmetic

11.2 (Pg. 837) # 7, 8, 11 (CW), 14, 15, 16, 29, 32, 35, 43, 44, 48, 49, 50, 61, 62, 63.

12.2 (P.798): 12, 11, 16, 17, 20, 19, 34, 36, 39, 48, 47, 51, 54, 53, 65, 66, 67

geometric — CW: 42, 49, 59, 60, 65

11.3 (Pg. 844) # 13, 14, 15, 20, 21, 31, 35, 38 (CW), 41, 44, 45, 51, 54, 59, 60, 73, 76, 64, 66, 68, 71

12.3 (P.806): 18, 17, 20, 24, 26, 35, 39, 42, 46, 48, 49, 56, 61, 67, 68, 81, 84, 72, 73, 76, 79

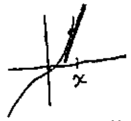
11.5 (Pg. 859) induction # 5, 13, 19, 23, 36.

12.5 (P.819) (CW: (11) 7, 14): 8, 16, 21, 25, 38

Bonus: $f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2)$

$= 3x^2$

$\cancel{x^3} + 3x^2h + 3xh^2 + h^3$



binomial

11.6 (Pg. 868) # 6, 7, 15, 17, 19, 23, 27, 28, 33, 37, 38, 40, 41, 43, 49.

12.6 (P.828): 9, 12, 20, 22, 16, 24, 31, 32, 37, 41, 42, 44, 46, 47, 53, 55, *55, 56

binomial random variable

Bonus Some Ch. 11 notes: Lesson #8

B1) Arith $x\ y\ z \Rightarrow y = \frac{x+z}{2}$, Geo $x\ y\ z \Rightarrow y = \sqrt{xz}$. Proof?

- B1) (+10 HW points) Use the technique showed in class involving linearity of summation to derive Pg. 858 818 formula #3 if you know formula #2 (or are) is true. Please do not use induction. Hint: Set up a collapsing sum with (i+1)^3 - i^3.
- B2) (+5 points) In the game "rock, paper, scissors", there are 3 rules: rock smashes scissors, scissors cut paper, paper covers rock. In the game "rock, paper, scissors, lizards, Spock", how many rules are needed? What if you have the game with n items: how many rules are needed? (another +5 HW points: What if you have a game with n+1 items: how many *more* rules are needed than the number of rules for the game with just n items? Relate this to the markers-eraser theorem)
- B3) (+5 HW points) Write nC2 in sigma notation.

11.1 (P. 830) # 5, 8, 11, 14, 15, 16, 21, 25, 27, 33,
seq & ser
37, 38, 41, 43, 45, 49, 55, 56, 58,
62, 63, 64, 65, 67, 73

11.2 (P. 837) # 7, 8, 11, 14, 15, 16, 29, 32, 35, 43, 44,
arith
48, 49, 50, 61, 62, 63.

5 ~ 8, 11, 15   19, 21, 27, 29, 32, 33, 35, 37, 39, 43
10 ~ 16,   46, 47, 48, 49, 53, 54, 58, 59, 61, 62
63

62) Falling Ball   ($g = -32$ ft/s^2)

$\Delta t_1 \Rightarrow$ 16ft $x(t) = \frac{1}{2}gt^2 = 16t^2$

2nd sec   48ft $d = 32$ $x(t) - x(t-1) = 16[(t)^2 - (t-1)^2]$

free fall as arithmetic series?

3rd sec   80ft

$= 16[\cancel{t^2} - \cancel{t^2} + 2t - 1]$

$\Delta x \approx dx = v(t)dt$

$= 32t - 16$

$= 16(2t-1)$

$\sum \Delta x \to \int v(t)dt$

11.3 (P. 844) # 13, 14, 15, 20, 21, 31, 35, ~~38~~, 41, 44, 45, 51, 54,
geo
59, 60, 73, 76

11.6 (P. 868) # 6, 7, , 15, 17, 19, 23, 27, 28, 33, 37, 38,
binomial
40, 41, 43, 49 (just like in class)

$f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$= 3x^2$

10.7
9.9 Systems of Inequalities

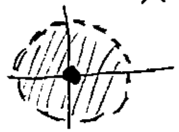
Graph $y > x^2$ $\{(x,y) : y > x^2\}$

① Graph the Equation(s)
② Test one point in each region. Shade

ex1 a) $x^2 + y^2 < 25$

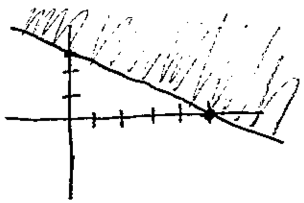
common sense

$0 < 25$
distance < 5
$100+100 \not< 25$

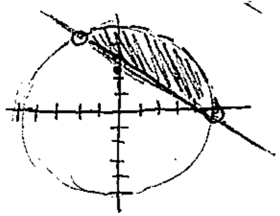


b) $x + 2y \geq 5$

$y \geq \frac{-x+5}{2} = \frac{-(x-5)}{2}$



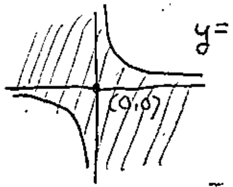
ex2 $\begin{cases} x^2+y^2 < 25 \\ x+2y \geq 5 \end{cases}$



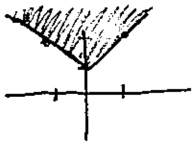
Aufmann & Baker P633

ex $xy \leq 1$

$y = \frac{1}{x}$

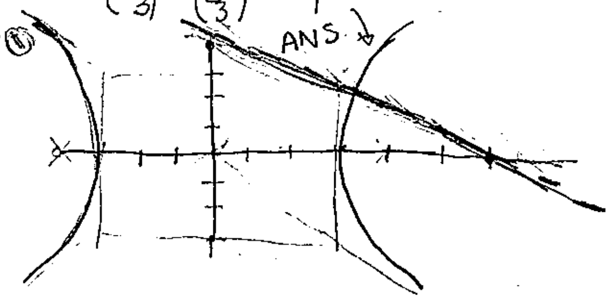


ex $y \geq |x| + 1$

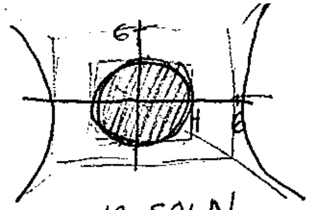


ex $x^2 - y^2 \leq 9 \leftarrow \left(\frac{x}{3}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$
$2x + 3y > 12$

$y > \frac{-2(x-6)}{3}$



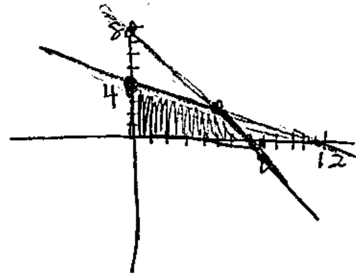
ex $\begin{cases} x^2+y^2 \leq 16 \\ x^2 - y^2 \geq 36 \end{cases}$



NO SOLN

ex3 $\begin{cases} x+3y \leq 12 \\ x+y \leq 8 \\ x \geq 0 \\ y \geq 0 \end{cases}$

$y \leq \frac{-(x-12)}{3}$
$y \leq -x+8$



Application: Feasible Regions
Linear Programming Pg 735 P716

		Cut	Sew	Profit
ex1	Oxfords = x	15 min	10 min	\$15
	Loafers = y	15	20 min	\$20
		8 hours	8 hours	

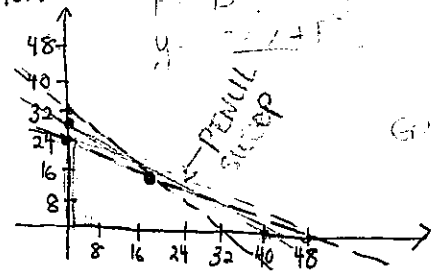
how many of each to maximize profit?

① constraints $\begin{cases} 15(x+y) \leq 480 \\ 10x + 20y \leq 480 \end{cases}$

HOURS
$\begin{cases} \frac{1}{4}(x+y) \leq 8 \\ \frac{1}{6}x + \frac{1}{3}y \leq 8 \\ x \geq 0 \\ y \geq 0 \end{cases}$
FEASIBLE REGION

MINUTES

OBJECTIVE FUNCTION $P(x,y) = 15x + 20y$



③ TEST VERTICES

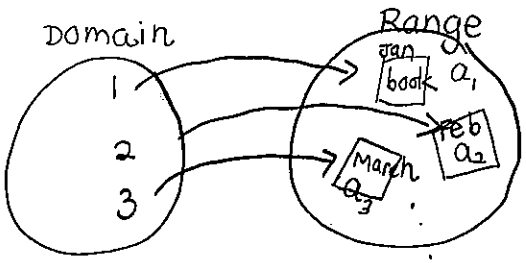
(x,y)	P
(0,24)	480
(16,16)	$\boxed{560}$
(32,0)	480
(0,0)	0

VERTICES: $x+y=32$
$x+2y=48$
$y=16 \rightarrow x=16$

Ch.11

Sequences & Series

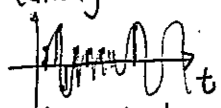
- Sequence $\{a_n\} = \{a_n = f(n) : n \in \mathbb{N}\}$
  function whose domain is countable (can map to $\mathbb{N}$)



simply: List of numbers

WHY?
- counting is fun! Patterns
- digital signal processing computer iterations
- Adding up pieces

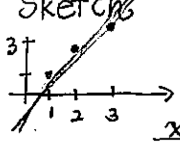
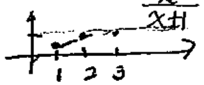
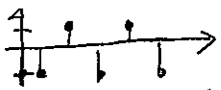
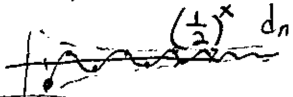
area? $\lim_{n\to\infty} A_n$ (n-sided polygon)



- discrete time
- can reconstruct whole signal!

$\boxed{ex}$ $2, 4, 6, 8, 10, \dots 2n$
$a_1, a_2, a_3, a_4, \dots \quad a_n$
↑ 1st term

$a_n = 2n$ ← index

$\boxed{ex1}$	1st 5 terms	Sketch	Converge?
$a_n = 2n-1$	$1, 3, 5, 7, 9$		$\lim_{n\to\infty}(2n-1)$ dne
$b_n = \frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$		$b_n \to 1$
$c_n = (-1)^n$	$-1, 1, -1, 1, -1$		c_n dne
$d_n = \left(\frac{-1}{2}\right)^n$	$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$		$d_n \to 0$

- a_n converges to L
  means $\lim_{n\to\infty} a_n = L$ $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.
  $|a_n - L| < \varepsilon$
  when $n \geq N$

$\boxed{ex2}$ There could be more than one way to describe a sequence. Use the most obvious

b) $\{-2, 4, -8, 16, -32, \dots\} = \{(-1)^n 2^n : n \in \mathbb{N}\}$
$\quad a_1 \ a_2 \ a_3$

$a_n = (-1)^n 2^n$

RECURSIVELY DEFINED SEQUENCE

$\boxed{ex3}$ $a_n = 3(a_{n-1} + 2)$ (RULE)

$n=1 \quad a_1 = 1$ (start)

$a_2 = 3(1+2) = 9$

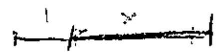
$a_3 = 3(9+2) = 33$

$a_4 = 3(33+2) = 105$

- note Pg 824: Large Prime Numbers → Computer security coding
- Sequence of Primes $(2, 3, 5, 7, 11, 13, 17, \dots)$ no explicit $f(n)$

$\boxed{ex4}$ Fibonacci Sequence

$F_n = F_{n-1} + F_{n-2}$ ← need 2 initial

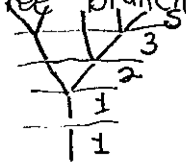


$F_1 = 1, \quad F_2 = 1$

$1, 1, 2, 3, 5, 8, 13, \dots$

Golden Ratio: $x = \frac{1+\sqrt{5}}{2}$
$\frac{1}{x} = \frac{x}{x+1} \quad \approx 1.618$

Tree branch, Shell, Flower, Fibonacci Quarterly on phenomena



$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$

See pg. 826: 1202 - use Hindu-Arabic numerals instead of Roman (required college degree to $\times \div$)

1299 - Florence outlawed 1, 2, 3 .. needed I, II, III for business

Series = sequence of partial sums (Fourier Series, Taylor Series)

$\{S_n\}$ $\quad \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$ add forever

does it converge?

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\boxed{S_n = \sum_{i=1}^{n} a_n}$

$S_n = a_1 + a_2 + \ldots + a_n$

n^{th} Partial Sum

ex Zeno of Elea Paradox

A rabbit can never go from A to B

- infinitely many halves to travel

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \stackrel{?}{=} 1$$

$\sum a_n$ "converges" to S means $\boxed{\lim_{n\to\infty} S_n = S}$

③ $\sum_{n=1}^{\infty} a_n$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

n ← end, $k=1$ ← START

k = index

ex ② $a_0 = 1$

$\sum_{k=0}^{5} a_k = 10$

$\sum_{k=1}^{5} b_k = 3$

$$\sum_{k=1}^{5}(3a_k - 2b_k) = 3\sum_{k=1}^{5} a_k - 2\sum_{k=1}^{5} b_k$$
$$= 3(10-1) - 2(3)$$
$$= 21$$

Σ practice

① ex7 b) $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

d) $\sum_{i=1}^{6} 2 = 2+2+2+2+2+2 = 12$

c) $\sum_{i=5}^{10} i = 5+6+7+8+9+10 = 45$

CHANGING index

ex8 a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \sum_{j=1}^{5} j^3 = \sum_{k=0}^{4} (k+1)^3$ $\quad j = k+1$, $k: 0 \sim 4$, $j: 1 \sim 5$

b) $\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77} = \sum_{n=3}^{77} \sqrt{n} = \sum_{k=1}^{75} \sqrt{k+2}$

$k = n-2 : 1 \sim 75$

$n = k+2 : 3 \sim 77$

③ ex5 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \stackrel{?}{=} 1$

① Seq of Partials
② Take Limit

series converges to 1

a) sequence of Partial sums

$S_1 = a_1 = \frac{1}{2}$

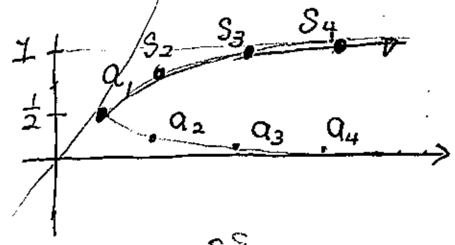
$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = a_1 + a_2 + a_3 = \frac{7}{8}$

$S_4 = \frac{15}{16}$

$\downarrow n \to \infty$

b) $\lim_{n\to\infty} S_n = \boxed{1}$



② Linear OPS Properties

① $\sum_{k=1}^{n} (a_k \pm b_k) = \sum a_k \pm \sum b_k$

② $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$

$(\Sigma a_k)(\Sigma b_k) \stackrel{?}{=} \cancel{\Sigma a_k b_k}$ OF COURSE NOT!

ex ③ $\sum_{n=1}^{\infty} (-1)^n$ does it converge?

$= -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

$\vdots$

$\lim_{n\to\infty} S_n = ?$ $\boxed{\text{dne}}$ diverges

④ ex6 Telescoping / Collapsing Sum

$a_n = \frac{1}{n} - \frac{1}{n+1}$ $\quad \sum_{n=1}^{\infty} a_n = ?$ does it converge

① $S_1 = 1 - \frac{1}{2}$

$S_2 = a_1 + a_2$

$S_N = \sum_{n=1}^{N} a_n = (1 - \cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}) + (\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}}) + \ldots + (\cancel{\tfrac{1}{N}} - \frac{1}{N+1})$

$= 1 - \frac{1}{N+1}$

② $\lim_{N\to\infty} S_N = \boxed{1}$

$1+2+3+4+\dots+100 = S$
$100+99+\dots+1 = S$
$101+101+\dots \quad N(N+1) = 2S$

12.2 Arithmetic Seq / Series

$a_1,\ a_1+d,\ a_1+2d,\ \dots\ a_1+(n-1)d$

- $a_n = a_1+(n-1)d$
- $S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}[a_1+(i-1)d] = na_1 - nd + d\left(\frac{n(n+1)}{2}\right)$

$\frac{n^2+n-n2}{2} = \frac{n(n-1)}{2}$

$= \boxed{na_1 + \frac{(n-1)n}{2}d}$ ✓

$= \frac{n}{2}[\underbrace{2a_1+(n-1)d}]$

$= \frac{n}{2}[a_1+a_n]$ ✓

ex2 $13, 7, \dots$ 300th?

$d=-6$

$13-6\times 299 = -1781$

ex3 $\dots, 52, \dots 92, \dots$ (11th, 19th) 1000th

$8d$

$52+8d=92$
$d=\frac{40}{8}=5$

$52+989\times 5 = 4997$

(47) $a_2=8,\ a_5=9.5,\ a_{15}= a_2+13\times\frac{1}{2}$

$8+3d=9.5$
$d=\frac{1}{2}$

$8+\frac{13}{2} = \frac{29}{2}$

ex5 $1+3+5+\dots(1+49\cdot 2) = \sum_{n=1}^{50}[1+(n-1)2]$

1st 50 odds

$= 50 + \cancel{2}\times\frac{49\times 50}{\cancel{2}}$ OR $\frac{[1+(1+49\times 2)]\times 50}{2}$

$= 2500$

ex6 50 rows

How many seats?

$30+32+34+\dots$

34, 32, 30

$a_n = 30+(n-1)2$

$\sum_{n=1}^{50} a_n = 50\times 30 + \frac{49\times 50}{\cancel{2}}\times\cancel{2}$

$= 3950$

ex7 $5, 7, 9, \dots$

add to when to get $S_N = 572$

$a_n = 5+(n-1)\times 2$

$S_N = 5N + \frac{(N-1)N}{\cancel{2}}\times\cancel{2} = 572$

$5N + N^2 - N = 572$

$N^2+4N-572=0$

$(N-22)(N+26)=0$

$N=\boxed{22}$ or $\cancel{-26}$

OR $572 = \frac{[5+(5+(N-1)2)]N}{2}$

$1144 = (10+2N-2)N$

$0 = N^2-4N-572$

(65) $15+18+21+\dots+\_\_ = 870$

$n=1 \quad n=2$

$a_n = 15+3\times(n-1)$

$N15 + 3\times\frac{(N-1)N}{2} = 870$

$30N + 3N^2 - 3N = 1740$

$N^2+9N-580=0$

$(N+29)(N-20)=0 \Rightarrow N=\boxed{20}$ or $\cancel{-29}$

2a

12.3 Geometric

$$\begin{cases} \overset{n=0}{a_0}, \overset{n=1}{ra_0}, r^2 a_0, r^3 a_0, \dots \\ a_n = a_0 r^{n-} \end{cases}$$

$$S_N = \sum_{n=0}^{N} a_n = a_0 \sum_{n=0}^{N} r^n = a_0(1+r+r^2+\dots r^N)$$

$$1+r+r^2+r^3+\dots = \begin{cases} S & |r|<1 \\ \text{dne} & |r|\ge 1 \end{cases}$$

$$S = 1+r+r^2+r^3+r^4+\dots+r^N$$
$$rS = r+r^3+r^4+r^5+\dots+r^N+r^{N+1}$$

$$S_N(1-r) = 1 - r^{N+1}$$

$$\boxed{S_N = \frac{1-r^{N+1}}{1-r}} = 1+r+r^2+r^3+\dots+r^N$$
finite STOP

$$S = \lim_{N\to\infty}\frac{1-r^{N+1}}{1-r} = \frac{1}{1-r}\left[1-\lim_{N\to\infty} r^{N+1}\right]$$

$$\begin{cases} \text{diverges if } |r|>1 \\ 0 \quad \text{if } |r|\le 1 \end{cases}$$

$$\boxed{\underset{1+r+r^2+\dots \text{ forever}}{S} = \begin{cases} \text{divg if } r>1 \\ \text{convg if } |r|\le 1 \end{cases}}$$

c) $1+(-\frac{1}{3})+\frac{1}{9}+\frac{-1}{27}+\frac{1}{81} = \frac{1-(-\frac{1}{3})^5}{1-(-\frac{1}{3})} = \frac{244}{243}\cdot\frac{3}{4} = \boxed{\frac{61}{81}}$ ... forever $|r|<1$ convg $\frac{1}{1-\frac{1}{3}} = \boxed{\frac{3}{4}}$

GEO $(-\frac{1}{3})^2+(-\frac{1}{3})^3+(-\frac{1}{3})^4$

d) $\ln 3+\ln 9+\ln 27+\ln 81+\dots = \ln 3+2\ln 3+3\ln 3+4\ln 3 = \frac{\ln 3+4\ln 3}{2}\cdot 4$
3^2, 3^3; $d=\ln 3$ arith
$= 2\ln(3\cdot 3^4) = \boxed{2\ln(243)}$

e) $\sum_{n=1}^{N}\ln\left(\frac{n}{n+1}\right) = [\ln 1-\cancel{\ln 2}]+[\cancel{\ln 2}-\cancel{\ln 3}]+\dots+[\cancel{\ln N}-\ln(N+1)]$
neither
$\sum_{1}^{\infty} = \lim_{N\to\infty} S_N = \ln\left(\frac{1}{N+1}\right) = \ln 0 = -\infty = \boxed{\text{dne}}$

ex2 $5, -15, 45, \dots$ 8th?
1st $n=0$, 2nd $n=1$, ... $n=7$

$$5\times r^7 = 5\times(-3)^7 = \boxed{-10{,}935}$$

ex3 $\dots, \frac{63}{4}, \dots \frac{1701}{32}, \dots$ 50th term?
3rd 4 5 6th; $50-6 = 44$ hops

$$\frac{1701}{32} = \frac{63}{4}\times r^3 \Rightarrow r = \frac{3}{2}$$

$$50^{th}: \frac{1701}{32}\times\left(\frac{3}{2}\right)^{44} = 2.97557\times 10^9$$

ex5
$$\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^k = 7\left(-\frac{2}{3}\right)+7\left(-\frac{2}{3}\right)^2+\dots+7\left(-\frac{2}{3}\right)^5$$
$$= 7\left(-\frac{2}{3}\right)\left[1+\left(-\frac{2}{3}\right)+\left(-\frac{2}{3}\right)^2+\dots+\left(-\frac{2}{3}\right)^4\right]$$
$$= -\frac{14}{3}\cdot\frac{1-(-\frac{2}{3})^5}{1-(-\frac{2}{3})}$$
$$= -\frac{14}{\cancel{3}}\cdot\frac{3^5+32}{5}\cdot\frac{\cancel{3}}{3^5} = \boxed{\frac{-770}{243}}$$

Add forever... $S = 7\frac{1}{1--\frac{2}{3}} = \boxed{\frac{21}{5}}$

Q Arithmetic or geometric? or neither Find the sum. If keep adding, will the series converge?

a) $2+4+8+16+32+\dots+1024$
$= 2(1+2+2^2+\dots+2^9) = 2\cdot\frac{1-2^{10}}{1-2} = \boxed{2046}$
Geo $|r|>1$ $\boxed{\text{divg}}$

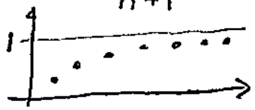
b) $\overset{N=1}{4}+6+8+10+\dots+1024 = \frac{(4+1024)}{2}\cdot 510 = \boxed{262140}$
arith $d=2$ $\boxed{\text{divg}}$ $|d|\ne 0$ arith will always $N=\frac{1022}{d}=511$

CALCULATOR

① seq

TI-89
2^{nd} math
3 List
1 seq

$u(n) = \frac{n}{n+1}$



seq (formula , variable, min, max, step) ⇒ List of seq
f(n)
no recursive

TI-83 GRAPH
$\boxed{seq}$ mode
$\boxed{Y=}$ nMin=1
\u(n)= n/(n+1)
$\boxed{Table}$ → shows list
$\boxed{GRAPH}$ . . . .

$\boxed{seq(\frac{n}{n+1}, n, 1, 10, 1)}$

② SERIES

TI-89
$\boxed{F3}$ calculus
4 Σ
Σ(formula, variable, min, max)
x^2 x 1 5

TI-83

$55 = \sum_{k=1}^{5} k^2$ sum(seq(K², K, 1, 5, 1)) step size

$\sum_{j=3}^{5} \frac{1}{j}$ $\boxed{\text{sum(seq(1/J, J, 3,5,1))}}$

. Arithmetic x y z
$y = \frac{x+z}{2}$ $\sqrt{xz}$

Arith. 2 5 8
$5 = \frac{10}{2}$

Geo 9 3 1
$3 = \sqrt{9 \times 1}$

1+2+3+ . . . +100
100+99+ . . . + 1
101+101+101+ . . . +101

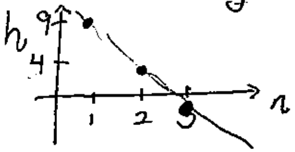
$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
1+2+...+n

$\boxed{11.2}$ ~11.3 – see SAT NOTES on arithmetic/geo seq
Arithmetic Sequence

$\boxed{ex1b}$ 9, 4, -1, -6, -11, . . . arithmetic or geometric?

$d = -5$
$a_n = 9 - (n-1)5$

c) Sketch



$\boxed{ex3}$ arithmetic
11^{th} term: 52
19^{th} : 92
1000^{th}?
1^{st}?
a_n = ? in 2 ways

$a_1, \ldots a_{11} \ldots a_{19}$
8 hops = 8d

$d = \frac{92-52}{8} = 5$

$a_{1000} = a_{19} + (1000-19)d$
$= 92 + 981 \times 5$
$= 92 + 4905$
$= 4997$

$a_1 = a_{11} - 10 \times d$
$= 52 - 10 \times 5$
$= 2$

$a_n = 2 + (n-1)5$

$a_n = 92 + (n-19)5 \quad \forall n \in \mathbb{N}$

$\boxed{ex5}$ Sum first 50 odd numbers

can easily memorize

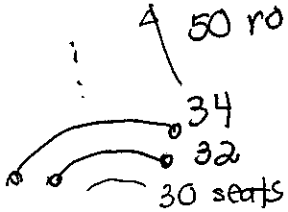
$S_N = N\left(\frac{a_1 + a_N}{2}\right)$
$= \frac{N}{2}(a_1 2 + (N-1)d)$

1, 3, 5, 7, 9, . . .
① a_1 a_2
$d = 2$
$a_{50} = 1 + 49 \times 2 = 99$
$S_{50} = 1+3+5+ \ldots +99 = 50\left(\frac{1+99}{2}\right) = 2500$

② OR $S_{50} = \sum_{n=1}^{50} [1 + (n-1)2]$
$= 50 + \frac{50 \times 51}{\cancel{2}} \times \cancel{2} - 50 \times 2$
$= 50(1 + 51 - 2) = 50 \times 50 = 2500$

why ... amphitheater

ex6 Amphitheater. How many seats?



① $30+32+34+\dots+[(49\times 2+30]$

$= 50\times\frac{30+49\times 2+30}{2}$

$= 50\times(30+49)$

$= 50\times 79 = \boxed{3950}$

② $\sum_{n=1}^{50}[30+(n-1)2] = 1500+\frac{50\times 51}{\cancel{2}}\times\cancel{2}-50\times 2$

$= 50(30+51-2) = 50\times 79$

$= \boxed{3950}$

ex7 $5, 7, 9, \dots$

Add how many terms to get 572?

Cannot use $N\left(\frac{a_1+a_N}{2}\right)$ directly

$S_N = \frac{N}{2}(a_1 2+(N-1)d) =$

$572 = \frac{N}{2}(10+(N-1)2)$

$572 = 5N+N(N-1)$

$= 5N+N^2-N$

$N^2+4N-572=0$

1   26
1   −22

$(N-22)(N+26)=0$

$\boxed{N=22}$

572
2   286
2   143
13   11

ex $1+\left(\frac{-1}{3}\right)+\frac{+1}{9}+\frac{-1}{27}+\dots+\frac{1}{81} = \frac{1-\left(\frac{-1}{3}\right)^5}{1-\left(\frac{-1}{3}\right)}$

$\left(\frac{-1}{3}\right)^4$

$+\dots$

convg? yes $|r|<1 \rightarrow \frac{1}{1-\frac{-1}{3}} = \frac{3}{4}$

ex $2+4+8+16+32+\dots+1024$

$= 2(1+2^2+\dots+2^9) = 2\frac{1-2^{10}}{1-2}$

$+\dots$ divg.

11.3 $1+r+r^2+r^3+\dots$ $n=0$ $n=1$

divg $|r|\geq 1$

convg $\boxed{\frac{1}{1-r}}$ $|r|<1$

must start with r

$S_N = 1+r+r^2+\dots+r^N = \boxed{\frac{1-r^{N+1}}{1-r}}\ \forall r$

$rS_N = \quad r+r^2+\dots+r^N+r^{N+1}$

$S_N(1-r) = 1 - \quad r^{N+1}$

11.3 Geometric Sequence   why ... probability ...

ex1 b) $2, -10, 50, -250, 1250, \dots$

$-5 \quad -5$

common ratio

$a_n = 2\times(-5)^{n-1} \quad n=1,2,3,\dots$

c) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$

$\frac{1}{3}$

$a_n = \left(\frac{1}{3}\right)^n \quad n=0,1,2,3,\dots$

ex2 $5, 15, 45, \dots$

eigth term?

$5\times 3^7 = 10935$

ex3 $3^{rd}: \frac{63}{4}$

$6^{th}: \frac{1701}{32}$

$5^{th}: ?$

$n=3$

$\frac{63}{4} = 7\left(\frac{3}{2}\right)^2$

$a_1 \quad a_2 \quad a_3 \xrightarrow{\times r} a_4 \xrightarrow{\times r} a_5 \xrightarrow{\times r} a_6$

$7 \quad 7\left(\frac{3}{2}\right)^1$

$\frac{1701}{32} = \frac{63}{4}\times r^3$

$\frac{27}{8} = r^3$

$r = \frac{3}{2}$

$a_5 = \frac{a_6}{r} = \frac{1701}{32}\times\frac{2}{3}$

$= \boxed{\frac{567}{16}}$

$a_0 = \frac{a_3}{r^2} = \frac{63}{4}\times\frac{4}{9} = 7$

$a_n = 7\left(\frac{3}{2}\right)^{n-1}$

$n = 0, 1, 2, \dots$

ex4 $1, 0.7, 0.49, 0.343, \dots$

$S_5 = \frac{1-0.7^5}{1-0.7}$ $1+0.7+(0.7)^2+\dots+(0.7)^4$

$= 2.7731$

ex5 $\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^k = 7\left(-\frac{2}{3} + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \dots + \left(-\frac{2}{3}\right)^5\right)$

$= 7\left(1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \dots + \left(-\frac{2}{3}\right)^5\right) - 7$

$= 7 \cdot \dfrac{1-\left(-\frac{2}{3}\right)^6}{1-\left(-\frac{2}{3}\right)} - 7$

$= 7 \cdot \frac{5}{3}\left(1 - \frac{64}{729}\right) - 7$

$7 \cdot \frac{5}{3}\,\frac{665}{729} - 7 = \frac{1966}{2187}$

k=1 k=2

$-\frac{14}{3} + \left(-\frac{14}{3}\right) \times \left(-\frac{2}{3}\right)^2$

$\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^k = 7\left[-\frac{2}{3} + \left(-\frac{2}{3}\right)^2 + \dots + \left(-\frac{2}{3}\right)^5\right]$

$= 7\left(-\frac{2}{3}\right)\left[1 + \left(-\frac{2}{3}\right) + \dots \left(-\frac{2}{3}\right)^4\right]$

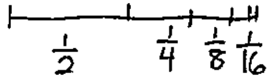
$= -\frac{14}{3} \cdot \dfrac{1-\left(-\frac{2}{3}\right)^5}{1 - -\frac{2}{3}} = -\frac{14}{3} \cdot \frac{3}{5}\left(1 + \frac{32}{243}\right)$

$= \boxed{-\frac{770}{243}}$

OR $7\left[1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \dots + \left(-\frac{2}{3}\right)^5\right] - 7$

$= 7 \cdot \dfrac{1-\left(-\frac{2}{3}\right)^6}{1 - -\frac{2}{3}} - 7 = -\frac{770}{243}$

ex Zeno of Elea



convg? $|r| < 1$ ✓

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots\right)$

$= \frac{1}{2} \cdot \dfrac{1}{1-\frac{1}{2}} = \frac{1}{2} \cdot 2 = 1$

ex6 $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \dots = 2\left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots\right) = \dfrac{2}{1-\frac{1}{5}} = \frac{5}{2}$

ex7 ✓ $2.3\overline{51} = S$

$2.351\,51\,51\,51$

$2351.515151\dots \quad 1000S$
$-\ \ 23.515151\dots \quad -\ 10S$

$2328 = 990S$

$S = \frac{2328}{990} = \boxed{\frac{388}{165}}$

✓Geometric method

$2 + \frac{3}{10} + \frac{51}{1000} + \frac{51}{100000} + \frac{51}{10000000} + \dots$

$= 2 + \frac{3}{10} + \frac{51}{1000}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \dots\right)$

$= 2 + \frac{3}{10} + \underbrace{\frac{51}{1000} \cdot \dfrac{1}{1-\frac{1}{100}}}_{\frac{51}{10\cancel{00}} \cdot \frac{\cancel{100}}{99}} =$

$= \frac{23}{10} + \frac{51}{990} = \boxed{\frac{388}{165}}$

Go to induction. Then Binomial

Uses * binomial r.v. = # success in n trials
* Physics SHM Pendulum (Maclaurin)

11.6 Binomial Thm

1. $_nC_r$ choose meaning
2. Pascal & formula. Examples 1~7
3. Proofs
   1. Binomial formula
   2. Pascal's Triangle
   3. $\sum i^2 = \frac{n(n+1)(2n+1)}{6}$ method see P.8

① A B C D E Counting Principle

$5 \times 4 \times 3 = {}_5P_3 = \frac{5!}{2!}$ ordered

ABC, BAC, BCA, ... overcounted $3!$

$\rightarrow {}_nC_r = {}_nP_r / r! = \frac{n!}{(n-r)!\,r!}$

$_5C_3 = \frac{_5P_3}{3!}$

② take 3 (5 books)

③ $\binom{n}{r} = \binom{n}{n-r}$

④ see P7

Committee $\rightarrow C_3$

P, VP, T $\rightarrow P_3$

$(a+b)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

ex $\boxed{(a+b)^5} = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & n=1\\
 & & & & 1 & & 1 & & & & n=2\\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \quad n=5
\end{array}
$$

$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \binom{n}{3}\ \binom{n}{r}$

$\binom{n-1}{r-1}\ \binom{n-1}{r}$

Proof

$$= \boxed{\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r}$$

I prefer $\binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \dots \sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}$

WHY (ancient China 1303) & Pascal later

$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\cdots(a+b)}_{n}$

must be leftovers multiplied are a. How many?

- n factors choose r to get a
- unordered

$= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$

do this for all exponent combos

Note

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$

unordered w/o replacement choose r from n

Pascal's Triangle: does it work?

Recursion Java

$$\boxed{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}}$$

$\cancel{\binom{n-1}{n-1-(r-1)}} = \binom{n-1}{n-r}$

$= \binom{n-1}{n-r} + \binom{n-1}{r} = \cancel{\binom{n-1}{n-r} + \binom{n-1}{n-1-r}}$

Prove Logically ??

$\frac{n!}{r!(n-r)!} \stackrel{?}{=} \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}$

$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$

$\Leftrightarrow \frac{n}{r(n-r)} = \frac{1}{1(n-r)} + \frac{1}{r \cdot 1}$ ✓

$\underbrace{}_{r+n-r}$

$\binom{5}{3} = \frac{5!}{3!2!} = \frac{5\times4\times3}{3\times2\times1}$

$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$

★ LOGIC ★

① 4 books gift basket of 3 books

4 books gift baskets with 2 books

5 books take 3

③ Use 2 old books & use the new book. This many ways

OR

② Now I have another book. How many baskets of 3 from the old 4 and the new 1?

- either keep the old way then add ways to put the new book into each old one by replacing one of the old ones $\binom{4}{2}$ out of 4 books, 2 kept, 1 replaced

$\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$

ex2 By Pascal

$(2-3x)^5 = 1(2)^0(-3x)^5 + 5(2)^1(-3x)^4 + 10(2)^2(-3x)^3 + 10(2)^3(-3x)^2 + 5(2)^4(-3x)^1 + 1(2)^5(-3x)^0$

$= -243x^5 + 810x^4 - 240x^3 + 720x^2 - 240x + 32$

ex3

a) $\binom{9}{4} = \frac{9!}{4!\,5!} = \frac{9\times8\times7\times6}{4\times3\times2} = 14\times9 = 90+36 = 126$

b) $\binom{100}{3} = \frac{100!}{3!\,97!} = \frac{100\times99\times98}{3\times2} = \boxed{161700}$

ex4 $(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

NOTE: binomial Maclaurin

$(x+1)^p = \sum_{r=0}^{\infty}\binom{p}{r}x^r 1 = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \dots$

$p \in \mathbb{R}$

$(x+1)^{\frac{1}{2}} = 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + \dots$

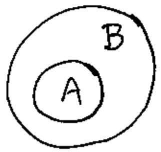
$\frac{\frac{1}{2}}{1!} = \frac{1}{2}$, $\quad \frac{\frac{1}{2}\times -\frac{1}{2}}{2!} = -\frac{1}{8}$, $\quad \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3\times2!} = \frac{1}{32}$

11.5 mathematical induction

Logic

- If A, then B $\quad A \Rightarrow B \quad B \not\Rightarrow A$

- Alice in Wonderland
  - current creates magnetic field
  - magnetic field does not create current unless changing



- A cat is an animal $\quad A \Rightarrow B$, cat $\Rightarrow$ animal
- An animal is a cat $\times \quad \not\Leftarrow$

$A \Rightarrow B$

- Contrapositive $B^c \Rightarrow A^c$
  - not an animal $\Rightarrow$ not a cat

$= A \cup B \cup C \subset D$

$D^c \Rightarrow A^c \cap B^c \cap C^c$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$|x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$,

$|x-c| < \delta$ but $\not\Rightarrow$ $|f(x) - L| > \varepsilon$

zic zoc

- If a zic is a zoc then a tic is a tac.
- If a tic is not a tac, then a zic is not a zoc.

- Proof by contradiction "reductio ad absurdum"

$A \Rightarrow B$ $\quad$ It's a rock $\Rightarrow$ must have weight. If not (weight), rock would float in air. Not a rock. $\rightarrow\leftarrow$

Assuming A is true but B is not, it leads to A not true

$\sqrt{2} \Rightarrow$ irrational

Suppose rational. Then $\sqrt{2} = a/b$ (lowest terms, one is odd) $\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$

a odd, a^2 odd, so a even, a^2 mult of 4, b^2 even, b even $\rightarrow\leftarrow$

"If you understand math, then you like it." $\quad A \Rightarrow B$

If you don't like it, you don't understand it. $B^c \Rightarrow A^c$

If you don't understand it, you could still not like it $A^c \not\Rightarrow B$

You could like it without understanding it. $B \not\Rightarrow A$

A ~~monotonically increasing~~ positive series: bounded above $\Updownarrow$ converge

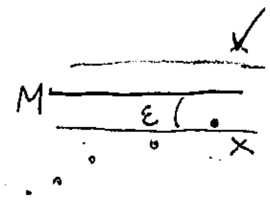
Proof ($\Downarrow$)

$S_N = \sum_{n=1}^{N} a_n$ is monotonically increasing

Let M be the least upper bound

If S_N does not converge, then for $\exists \varepsilon > 0$ and $K \in \mathbb{N}$

s.t. $|S_K - M| > \varepsilon$

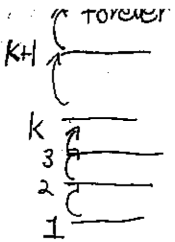


$\cancel{S_K < \underbrace{M - \varepsilon}_{\text{not an upper bound}}}$ OR $S_K > M + \varepsilon$

so M is not an upper bound $\rightarrow\leftarrow$

Show Domino effect

- P(1) is true — 1st domino falls right
- Assume P(k) is true — If one falls, neighbor will fall
- Show P(k+1) is true



CS & recursion

Cohen's horse

$\boxed{\text{ex1}}$

$\boxed{\text{ex3}}$ Prove $4n < 2^n \quad \forall n \geq 5$

- $n=5 \quad 20 < 2^5$ ✓
- Assume $4n < 2^n$

(Show $4(n+1) \overset{?}{<} 2^{n+1}$)

$4n < 2^n$

$4(n+1) = 4n+4$

hypothesis $< 2^n + 4 = 2^n + 2^2$

$< 2^n + 2^n \quad \because n > 2$

$= 2^n(2)$

$= 2^{n+1}$

like $\boxed{\text{ex2}}$ Prove (Bonus: by not induction) after Binomial introduced)

$1+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$n=1: \quad 1 = \frac{1\times(1+1)(2\times1+1)}{6} = \frac{2\times3}{6}$ ✓

Suppose true for n.

(Show true for $n+1$.

i.e. $1+2^2+\dots+(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$ assume

$\underbrace{1+2^2+3^2+\dots}+(n+1)^2$

$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}$

$= \frac{1}{6}(n+1)[n(2n+1)+6(n+1)]$

$2n^2+n+6n+6$

$2n^2+7n+6$

$= \frac{1}{6}(n+1)(n+2)(2n+3)$ ✓

⑥ $1\cdot3+2\cdot4+3\cdot5+\dots+n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Proof:

$n=1 \quad 1\cdot3 = \frac{1\times2\times9}{6}$ ✓

$n=2 \quad \underset{k=1}{1\cdot3} + \underset{k=2}{2\times4} = \frac{2\times3\times11}{6} = 11$ ✓

$$\sum_{k=1}^{n} k(k+2) = \frac{n(n+1)(2n+7)}{6}$$

$$\left(\text{Show } \sum_{k=1}^{n+1} k(k+2) \overset{?}{=} \frac{(n+1)(n+2)(2n+9)}{6}\right)$$

$$\sum_{k=1}^{n+1} k(k+2) = \sum_{k=1}^{n} k(k+2) + (n+1)(n+3)$$

$$= \frac{n(n+1)(2n+7)}{6} + \frac{6(n+1)(n+3)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6n+18)}{6}$$

$$= \frac{(n+1)(2n^2+13n+18)}{6}$$

$$= \frac{(n+1)(2n+9)(n+2)}{6}$$ ✓

$\boxed{\text{ex1}}$ Add the odd numbers

1

$1+3=4 \quad 2^2$

$1+3+5=9 \quad 3^2$

$1+3+5+7=16 \quad 4^2$

$$\sum_{i=1}^{n}(2i-1) \overset{??}{=} n^2$$

Proof:

- $n=1: \quad 1=1^2$ ✓
- Suppose $\sum_{i=1}^{n}(2i-1) = n^2$ ✓
- Show $\underbrace{\sum_{i=1}^{n+1}(2i-1)} = (n+1)^2$

$n^2 + 2(n+1) - 1 = n^2+2n+1$

$= (n+1)^2$ ✓

HW #5, 13, 19, 23, 36

(P.859)

ex5 Use Binomial Thm to expand $(\sqrt{x}+1)^8$

$$= \binom{8}{0}(\sqrt{x})^{\cancel{0}}(-1)^8 + \binom{8}{1}(\sqrt{x})^{1}(-1)^7 + \binom{8}{2}(\sqrt{x})^{2}(-1)^6 + \binom{8}{3}(\sqrt{x})^3(-1)^5 + \binom{8}{4}(\sqrt{x})^4(-1)^4 + \binom{8}{5}(\sqrt{x})^5(-1)^3 + \binom{8}{6}(\sqrt{x})^6(-1)^2 + \binom{8}{7}(\sqrt{x})^7(-1)^1 + \binom{8}{8}(\sqrt{x})^8(-1)^0$$

$\binom{8}{0}=1$, $\binom{8}{1}=8$, $\frac{8!}{2!6!} = \frac{8\times7}{2} = 28$, $\frac{8!}{3!5!} = \frac{8\cdot7\cdot6}{3\cdot2} = 56$, $\frac{8!}{4!4!} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2} = 70$, $\binom{8}{5}=3$, $\binom{8}{6}=2$, $\binom{8}{7}=1$, $\binom{8}{8}=0$

$$= 1-8x^{\frac{1}{2}} + 28x + 56x^{3/2} + 70x^2 + 56x^{5/2} + 28x^3 + 8x^{7/2} + x^4$$

ex6 $(2x+y)^{20}$

The term containing x^5?

$$\binom{20}{5}(2x)^5(y)^{15} = \frac{20!}{5!\,15!}\,32x^5y^{15}$$

$$\frac{20\cdot19\cdot18\cdot17\cdot16}{5\cdot4\cdot3\cdot2} \Rightarrow 76\times17\times12 = \boxed{496128x^5y^{15}}$$

ex7 coefficient of x^8 in $\left(x^2+\frac{1}{x}\right)^{10}$

$8 = 2r-10+r \Rightarrow 3r=18 \Rightarrow r=6$

$(x^2)^r\,(x^{-1})^{10-r}$

$$\binom{10}{6}(x^2)^6(x^{-1})^4 = \boxed{210}x^{12-4}$$

$$\frac{10!}{6!\,4!} = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2} = 210$$

• factorial algebra (from SAT notes) Barron's

Simplify

① $\frac{(n+4)!}{(n+7)!} \Rightarrow \frac{1}{(n+5)(n+6)(n+7)}$

② $\frac{n^n}{n!} = \frac{nx}{(n-1)!}$ $\quad x=?$ $\quad \frac{n^{n-1}}{n} = n^{n-2}$

$\frac{n^{n-1}}{n!} = \frac{\cancel{n}x}{\cancel{(n-1)!}}$

③ $\frac{5!}{6!-5!} = \ldots \frac{5!}{5!(6-1)} = \frac{\cancel{5!}}{\cancel{5!}\,5} = \frac{1}{5}$

④ $\frac{x!(x+1)!}{(x-1)!} = \ldots x(x+1)!$

$(2x+y)^{20} \rightarrow y^2$

$\binom{20}{2}(2x)^{18}y^2$

$\frac{20!}{2!\,18!}\,2^{18}\,x^{18}y^2$

$\frac{20\cdot19}{2}$

$190\times2^{18}\,x^{18}y^2$

(P. 818)

① $\sum_{i=1}^{n} 1 = n$ ② $\sum_{i=1}^{n} i = 1+2+3+\dots+n = \frac{n(n+1)}{2}$ ← $\begin{array}{r} 1+2+\dots+100 \\ 100+99+\dots+1 \\ \hline 101+101+\dots+101 \end{array} = 100\times 101$

③ $\sum_{i=1}^{n} i^{\boxed{2}} = 1+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ ← §12.5 Proved by induction
← Quiz Bonus Demo Here

④ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ ← HW Bonus

∘ Prove ③ (not by induction)

$\underbrace{\sum_{i=1}^{n} (i+1)^{\boxed{3}} - i^3}_{\text{collapsing sum}} = \sum_{i=1}^{n} \cancel{i^3} + 3i^2 + 3i + 1 - \cancel{i^3}$ goes away

LHS $= (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + \dots + [(n+1)^3 - \cancel{n^3}]$

$= -1 + (n+1)^3 = \cancel{1} + n^3 + 3n^2 + 3n + \cancel{1}$

RHS $= 3\left[\sum_{i=1}^{n} i^2\right] + 3\sum i + \sum 1$

$= 3 \cdot \left[\underset{\text{solve for}}{S_n}\right] + 3\frac{n(n+1)}{2} + n$

$\Rightarrow n^3 + 3n^2 + 3n = 3S_n + \frac{3}{2}(n^2+n) + n$

$2n^3 + 6n^2 + 6n = 6S_n + 3n^2 + \underline{3n + 2n}$

$2n^3 + 3n^2 + n = 6S_n$

$n(2n+1)(n+1) = 6S_n$ ✓

Prove ④

$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \end{array}$

$\sum (i+1)^4 - i^4 = \sum \cancel{i^4} + 4i^3 + 6i^2 + 4i + 1 - \cancel{i^4}$

$\overbrace{(\cancel{2^4} - 1^4) + (\cancel{3^4} - \cancel{2^4}) + \dots + [(n+1)^4 - \cancel{n^4}]}^{\Sigma}$

$-1 + (n+1)^4 = 4\sum i^3 + 6\sum i^2 + 4\sum i + \sum 1$

$\cancel{1} + n^4 + 4n^3 + 6n^2 + 4n + \cancel{1} = 4\sum i^3 + \cancel{6}\frac{n(n+1)(2n+1)}{\cancel{6}} + \overset{2}{\cancel{4}}\frac{n(n+1)}{\cancel{2}} + n$

$n[2n^2 + 3n + 1 + 2n + 2 + 1]$

$= 4\sum i^3 + 2n^3 + 5n^2 + 4n$

$n^4 + 2n^3 + n^2 = 4\sum i^3$

$n^2 \underbrace{(n^2 + 2n + 1)}_{(n+1)^2}$

$\left[\frac{n(n+1)}{2}\right]^2 = \sum i^3$

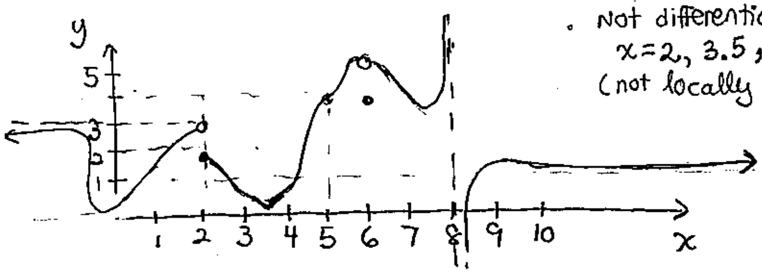
Ch13 Intro to Calculus

11.1 ~~19, 21~~, ~~35~~, ~~39~~

11.2 ~~12~~, ~~13~~,

11.3 ~~16, 19, 20~~

11.4 ~summary



- Not differentiable at $x = 2, 3.5, 6, 8$ (not locally linear)

$\boxed{\text{differentiable} \Rightarrow \text{continuous} \atop \not\Leftarrow}$

$f(x) = \frac{\square}{0} = \frac{-2}{(x-8)}$

$\lim_{x\to 2^-} f(x) = 3$

$\lim_{x\to 2^+} f(x) = 2$

$\lim_{x\to 2} f(x) = \text{dne}$

$\lim_{x\to 5} f(x) = 4$

$\lim_{x\to 6} f(x) = 5$

when x is near 6, y (f) is near 5

V.A.

$\lim_{x\to 8^-} f(x) = \infty$

$\lim_{x\to 8^+} f(x) = -\infty$

$\lim_{x\to\infty} f(x) = \text{dne}$

V.A.

$\lim_{x\to\infty} f(x) = 1$

$\lim_{x\to -\infty} f(x) = 3$

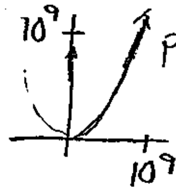
$x\to\infty$ Take leading terms

$\lim_{x\to\infty} \frac{\boxed{5x^3}+2x}{\boxed{5x^3}-1} = 1 \Rightarrow y=1$ is H.A.

$\lim_{x\to -\infty} \frac{\boxed{12x^5}-1}{\boxed{4x^5}+x} = 3 \Rightarrow y=3$ is H.A.

$\lim_{x\to\infty} \frac{\boxed{x^5}-1}{\boxed{3x^7}+2x+1} = 0 \Rightarrow y=0$ is H.A.

$\lim_{x\to\infty} \frac{2x^3+1}{2x-1} \sim x^2$

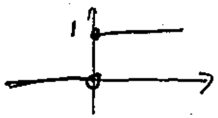


parabolic, no H.A.

Not continuous at $x=2$ (left $\neq$ right limits), $x=8$ limit dne & $f(8)$ dne, $x=6$ limit exists but $\neq f(6)$

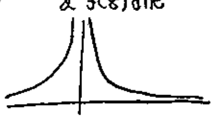
discontinuous at $x=0$ since $f(0)$ dne & limit dne

<u>13.1</u>

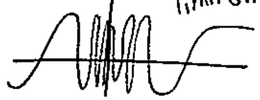


$H(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$

$\lim_{t\to 0} H(t)$ dne

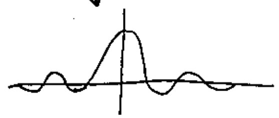


$\lim_{x\to 0} \frac{1}{x^2}$ dne



$\lim_{x\to 0} \sin\frac{\pi}{x}$

dne



$\lim_{x\to 0} \frac{\sin x}{x} = 1$

even though $f(0)$ dne

<u>13.2</u> Algebraically

$x\to\infty$. Take leading terms

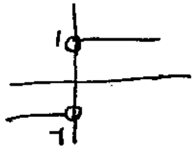
$x\to c$. Plug. If $\frac{0}{0}$, cancel. Then plug

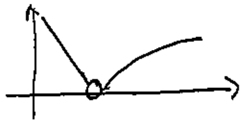
$\lim_{x\to 5} 2 = 2$

$\lim_{x\to 5} (2x^2 - 3x + 4) = 2\times 25 - 3\times 5 + 4 = 39$

$\lim_{x\to 1} \frac{x^2-1}{x-1} \overset{0/0}{=} \lim_{x\to 1} (x+1) = 2$

- $\lim_{x\to0}|x| = 0$
- $\lim_{x\to0}\frac{|x|}{x} = \begin{cases}\lim_{x\to0^+}\frac{x}{x}=1\\ \lim_{x\to0^-}\frac{-x}{x}=-1\end{cases}$ dne



- 

$f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4\\ 8-2x & \text{if } x<4\end{cases}$

$\lim_{x\to4} f(x) = ?$

$\lim_{x\to4^+}\sqrt{x-4} = 0$

$\lim_{x\to4^-} 8-2x = 8-2\cdot4 = 0$ ✓

MLT

$\lim_{x\to c}\left(f(x) \substack{\pm\\ \times\\ \div} g(x)\right)$

$= \left[\lim_{x\to c} f(x)\right] \substack{\pm\\ \times\\ \div} \left[\lim_{x\to c} g(x)\right]$

13.3 Continuous & others

- f is continuous at c means you can draw w/o picking up your pencil

$\boxed{\underset{(2)}{\lim_{x\to c} f(x)} \underset{(3)}{=} \underset{(1)}{f(c)}}$

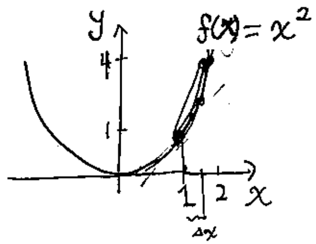
- See p1
- Continuous functions: polynomials, trig where defined

Derivatives

- Show TI
- Zoom in, looks straight

★ "Differentiable" = locally linear

See p. 1



- Slope of $\boxed{\text{secant}}$ from $x=1$ to $x=2$

$m_{sec} = \frac{f(2)-f(1)}{2-1} = \frac{\Delta f}{\Delta x}$ = $\boxed{\text{avg rate}}$ of change of f wrt x on $[1,2]$

- Slope of secant from $x=1$ to $x=1+\Delta x$

$m_{sec} = \frac{f(1+\Delta x)-f(1)}{\Delta x} = \frac{\Delta f}{\Delta x}$

- slope of $\boxed{\text{tangent}}$ at $x=1$

$\left.\frac{df}{dx}\right|_{x=1} = f'(1)$ = limit of secant slopes = $\boxed{\text{instantaneous}}$ rate of change of f at $x=1$

$= \lim_{\Delta x\to0}\frac{\Delta f}{\Delta x}$

$b = 1+\Delta x$, $\Delta x\to0$, $b\to1$

$= \boxed{\lim_{\Delta x\to0}\frac{f(1+\Delta x)-f(1)}{\Delta x}} = \lim_{b\to1}\frac{f(b)-f(1)}{b-1}$

a) $\lim_{\Delta x\to0}\frac{(1+\Delta x)^2-1^2}{\Delta x} \overset{0/0}{=} \lim_{\Delta x\to0}\frac{\cancel{x}+2\Delta x+(\Delta x)^2-\cancel{x}}{\Delta x}$

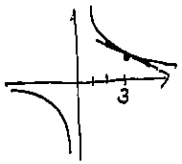
$= \lim_{\Delta x\to0}(2+\Delta x) = \boxed{2}$

b) $\lim_{b\to1}\frac{(b)^2-1^2}{b-1} = \lim_{b\to1}\frac{\cancel{(b-1)}(b+1)}{\cancel{b-1}} = 1+1 = \boxed{2}$

Eqn of tangent line $\boxed{y = 2(x-1)+1}$

$f'(1) = 2x'\Big|_{x=1} = 2\cdot1$ ← Check $f(x)=x^3$, $f'(x)=3x^2$ etc...

ex1 $y = \frac{3}{x}$ eqn of line tangent at point $(3, 1)$



$y = m(x-3)+1$

$m = f'(3)$

$= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

$= \lim_{b \to 3} \frac{f(b) - f(3)}{b-3} \overset{\frac{0}{0}}{=} \lim_{b \to 3} \frac{\frac{3}{b} - \frac{3}{3}}{b-3}$

$= \lim_{b \to 3} \frac{1}{b} \frac{3-b}{b-3} = \lim_{b \to 3} \left(-\frac{1}{b}\right) = \boxed{-\frac{1}{3}}$

$\boxed{y = -\frac{1}{3}(x-3)+1}$

Check $f'(x) = -3x^{-2}$

$f(x) = 3x^{-1}$ $\quad f'(3) = -3(3)^{-2} = -\frac{3}{9} = -\frac{1}{3}$

cw

ex4 Find the equation of the tangent line to $y = \sqrt{x}$ at point $(9, 3)$

$y = m(x-9)+3 = \boxed{\frac{1}{6}(x-9)+3}$

$m = f'(9) = \lim_{h \to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h} \overset{\frac{0}{0}}{\cdot} \frac{\sqrt{9+h}+\sqrt{9}}{\sqrt{9+h}+\sqrt{9}}$

$= \lim_{h \to 0} \frac{\cancel{9}+\cancel{h}-\cancel{9}}{\cancel{h}(\sqrt{9+h}+\sqrt{9})} = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

$m = \lim_{b \to 9} \frac{\sqrt{b} - \sqrt{9}}{b-9} \overset{\frac{0}{0}}{=} \lim_{b \to 9} \frac{\sqrt{b}-3}{b-9} \cdot \frac{\sqrt{b}+3}{\sqrt{b}+3} = \lim_{b \to 9} \frac{b-9}{(b-9)(\sqrt{b}+3)}$

$= \frac{1}{3+3} = \boxed{\frac{1}{6}}$

Check: $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow f'(9) = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

13.4 See P. 1.

$\lim_{x \to \infty} f(x) = 5 \Leftrightarrow y=5$ is H.A.

$\lim_{x \to -\infty} f(x) = 3 \Leftrightarrow y=3$ is HA

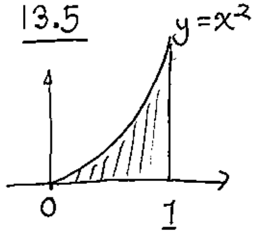
$\lim_{x \to \infty} \frac{5x^2-1}{3x^2+1} = \underline{5/3}$ $\qquad \lim_{x \to -\infty} e^x = \underline{0}$

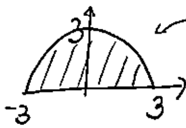
$\lim_{x \to \infty} \frac{1}{x} = \underline{0}$ $\qquad \lim_{x \to \infty} \sin x = \underline{\text{dne}}$

$\lim_{n \to \infty} \frac{n}{n+1} = \underline{1}$ $\qquad \lim_{n \to \infty} (-1)^n = \underline{\text{dne}}$

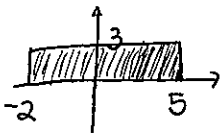
$\lim_{n \to \infty} \frac{15}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{15 \times 2}{6} = \underline{5}$

13.5



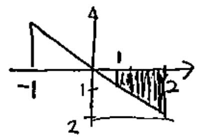


Notation $\int_{-3}^{3} f(x)\,dx = \frac{1}{2}\pi 3^2$



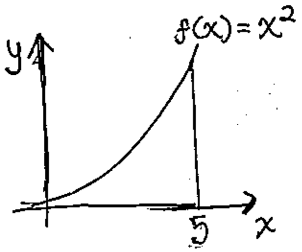
$\int_{-2}^{5} 3\,dx = 3 \times 7 = 21$

Definite Integral = Signed Area



$\int_{1}^{2} (-x)\,dx = -\frac{1}{2}(1+2) \times 1$

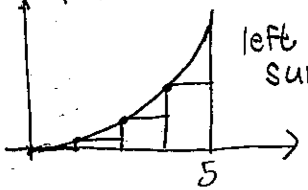
$= -\frac{3}{2}$



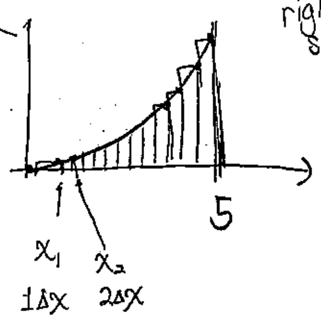
ex1~3

$$\int_0^5 x^2\,dx = ?$$

right Riemann sum $\left[f(\tfrac{5}{4})+f(\tfrac{5}{2})+f(\tfrac{15}{4})+f(5)\right]\times\frac{15}{4}$



left Riemann sum $\left[f(0)+f(\tfrac{5}{4})+f(\tfrac{5}{2})+f(\tfrac{15}{4})\right]\times\frac{15}{4}$



right Riemann sum

n pieces

$$\Delta x = \frac{5}{n}$$

$$x_i = 0 + i\Delta x$$

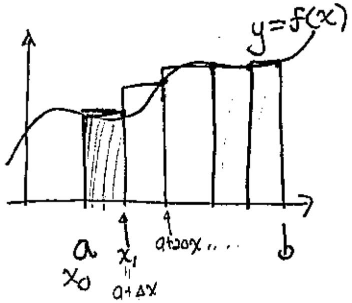
$$A_n = [f(x_1)+f(x_2)+\ldots+f(x_n)]\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x = \left[\left(\frac{5}{n}\right)^2+\left(2\frac{5}{n}\right)^2+\left(3\frac{5}{n}\right)^2+\ldots\left(n\cdot\frac{5}{n}\right)^2\right]\cdot\frac{5}{n} = \left(\frac{5}{n}\right)^3\left[1^2+2^2+3^2+\ldots+n^2\right]$$

$$= \left(\frac{5}{n}\right)^3\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} A_n = \frac{5^3}{6}\cdot 2 = \frac{5^3}{3} = \boxed{\frac{125}{3}} \approx 41.7$$

Check: $\int_0^5 x^2\,dx = \left[\frac{x^3}{3}\right]_0^5 = \frac{125}{3} - 0$



$$\Delta x = \frac{b-a}{n}$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\sum_{k=1}^{n} \overbrace{f(x_k)}^{③\text{height}}\,\Delta x_k}$$

$a+k\Delta x$, ② right endpoint

$\Delta x = \frac{b-a}{n}$ ① width

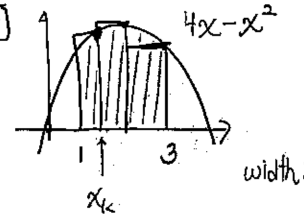
$$\sum_1^n i = \frac{n(n+1)}{2}$$

$$\sum_1^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_1^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

explain like that

ex4



$$\int_1^3 (4x-x^2)\,dx = ?$$

width: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

height: $f(x_k) = 4\left(1+\frac{2k}{n}\right) - \left(1+\frac{2k}{n}\right)^2$

right: $x_k = 1 + k\Delta x = 1 + \frac{2k}{n}$

1 box: $f(x_k)\Delta x$

all n boxes: $A_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left[4\left(1+\frac{2k}{n}\right)-\left(1+\frac{2k}{n}\right)^2\right]\frac{2}{n}$

$$= \frac{2}{n}\sum_{k=1}^{n}\left[4+\frac{8k}{n}-\left[1+\frac{4k}{n}+\frac{4k^2}{n^2}\right]\right]$$

$$= \frac{2}{n}\sum_{k=1}^{n}\left[3+\frac{4k}{n}-\frac{4k^2}{n^2}\right] = \frac{2}{n}\left[3n+\frac{4}{n}\,\frac{n(n+1)}{2}-\frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$\Rightarrow \lim_{n\to\infty} A_n = 2\left[3+\frac{4}{n^2}\,\frac{n^2}{2}-\frac{4\cdot 2n^3}{6n^3}\right]$$

$$2\left[3+2-\frac{4}{3}\right] = \boxed{\frac{22}{3}}$$