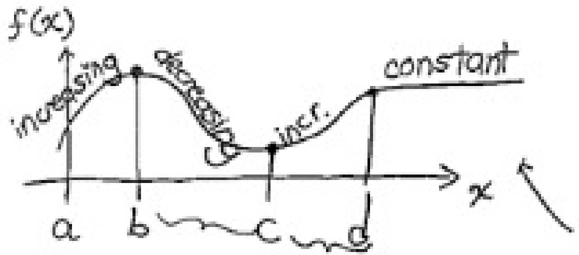
Calculus AB

Ch.4 Derivative Applications

4.1 (pg. 232) Textbook Examples

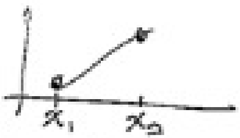
#3, 5, 6, 7, 9, ..., 13, 19, 21, 29, 31, 45, 55

Bonus: 56, 58



$f'(x) > 0$ on (a,b) $\quad$ $f'(x) < 0$ on (b,c) $\quad$ $f'(x) > 0$ on (c,d) $\quad$ $f'(x) = 0$ on (d,∞)

$f'(b) = f'(c) = 0$



$f(x)$

Increasing means $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Decreasing means $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Strictly monotonic means f is either increasing or decreasing

① First Derivative & Monotonicity

Thm (Monotonicity) f is continuous and differentiable on I

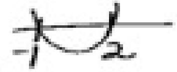
(i) $f'(x) > 0 \ \forall x \in I \Rightarrow f$ increasing on I

(ii) $f'(x) < 0 \ \forall x \in I \Rightarrow f$ decreasing on I

ex1 $f(x) = 2x^3 - 3x^2 - 12x + 7$

Find where f is increasing? decreasing?

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$
$= 6(x-2)(x+1)$



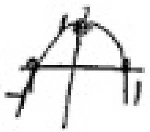
$f'(x) > 0$ on $(-\infty, -1) \cup (2, \infty)$ f increasing

$f'(x) < 0$ on $(-1, 2)$ $\therefore f$ decreasing

ex2 $g(x) = \dfrac{x}{1+x^2}$

Where is g increasing? decreasing?

$$g'(x) = \frac{(1+x^2) - x\,2x}{(1+x^2)^2} = \frac{-x^2+1}{(1+x^2)^2}$$

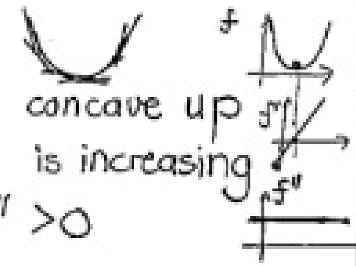


$$= \frac{(1-x)(1+x)}{(1+x^2)^2}$$

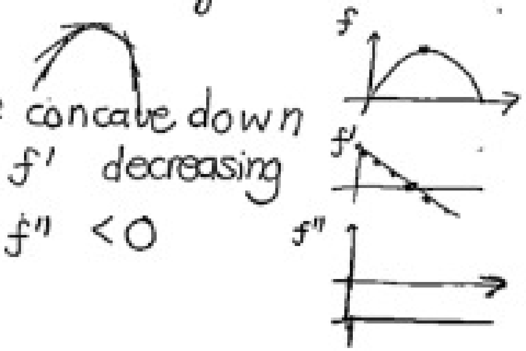
$g'(x) > 0$ on $(-1,1)$ $\therefore g$ increasing on $(-1,1)$

$g'(x) < 0$ on $(-\infty,-1) \cup (1,\infty)$ $\therefore g$ decreasing there.

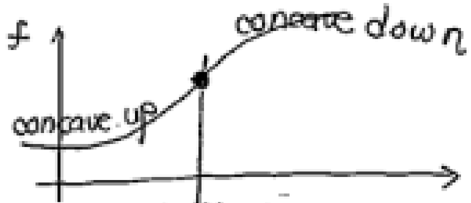
② 2nd Derivative & Concavity



f concave up
f' is increasing
$f'' > 0$



f concave down
f' decreasing
$f'' < 0$



Thm Concavity Thm

f is twice differentiable on open interval I

(i) $f''(x) > 0 \quad \forall x \in I \Rightarrow f$ is concave up on I

(ii) $f''(x) < 0 \quad \forall x \in I \Rightarrow f$ is concave down on I

$\boxed{\text{ex3}}$ $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$

where is f increasing: on $(-\infty,-1) \cup (3,\infty)$

decreasing $(-1,3)$

concave up $(1,\infty)$

concave down $(-\infty,1)$

$f'(x) = x^2 - 2x - 3 = (x-3)(x+1)$ -1 3

$f''(x) = 2x - 2 = 2(x-1)$ 1

10min

$\boxed{\text{ex4}}$ $g(x) = \dfrac{x}{1+x^2}$

a) Where is g concave up? $(-\sqrt{3},0) \cup (\sqrt{3},\infty)$

concave down? $(-\infty,-\sqrt{3}) \cup (0,\sqrt{3})$

b) sketch $g(x)$

$$g'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{-x^2+1}{(1+x^2)^2}$$

-1 1

$$g''(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)\,2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)\left[(1+x^2)(-2x) - 4x(1-x^2)\right]}{(1+x^2)^4}$$

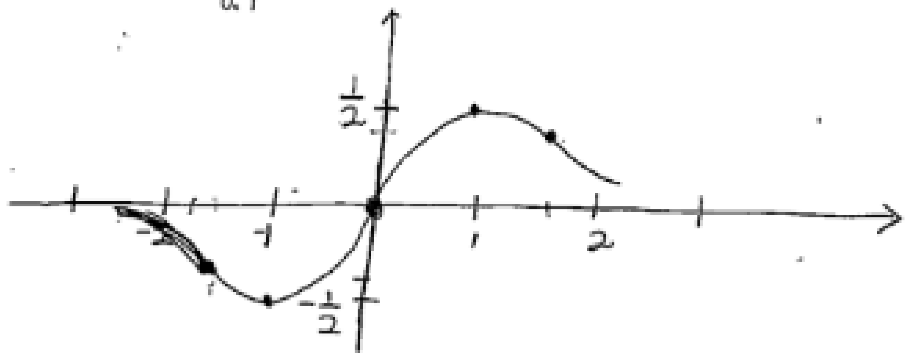
$$= \frac{-2x-2x^3-4x+4x^3}{(1+x^2)^3} = \frac{2x^3-6x}{(1+x^2)^3}$$

$$= \frac{+2x(x^2-3)}{\underbrace{(1+x^2)^3}_{>0}} \qquad 2x(x-\sqrt{3})(x+\sqrt{3})$$

$-$ $-\sqrt{3}$ $+$ 0 $-$ $\sqrt{3}$ $+$

Sketch:

x		$-\sqrt{3}$ (-1.73)		-1		0		1		$\sqrt{3}$	
$g'(x)$	$-$		$-$	0	$+$		$+$	0	$-$	0	$-$
$g''(x)$	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
$g(x)$		$-\frac{\sqrt{3}}{4}$ (0.4)		$-\frac{1}{2}$		0		$\frac{1}{2}$		$\frac{\sqrt{3}}{4}$	



c) Inflection Points? $(-\sqrt{3}, -\sqrt{3}/4)$

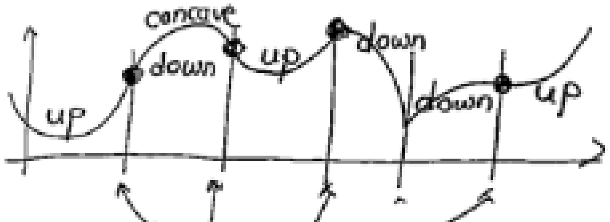
$(0,0)$

$(\sqrt{3}, \sqrt{3}/4)$

10min

Inflection Point $(c, f(c))$

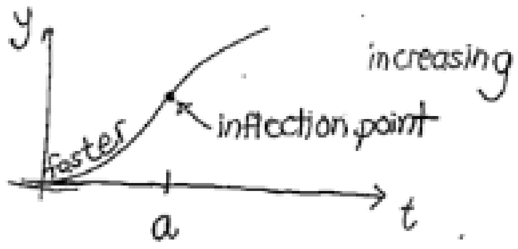
f is concave up on one side, concave down on other



inflection points

$\boxed{f''(x) = 0 \text{ or } f''(x) \text{ dne}}$ may be inflection points

★First try then see candidates



(ex) $f(x) = x^4$

$f''(0) = 0$ but $(0, 0)$ is not an inflection point

(ex) $F(x) = x^{1/3} + 2$

$F'(x) = \frac{1}{3} x^{-2/3}$

$F''(x) = -\frac{2}{9} x^{-5/3} = -\frac{2}{9x^{5/3}}$ $\begin{cases} \neq 0 \\ \text{dne when } x = 0 \end{cases}$

Is $x=0$ an inflection point?

x		0	
$F''(x)$	+	dne	−
		yes	

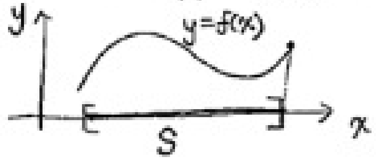
AP Calculus C 11/24/2007 Tue.

- due Tue. 11/30
  - HW #8 { §7.5, 11.5, 7.7 } §7.4 - #44, 45; Thm A, Thm B to prove
  - HW Corrections
- due today HW#7 §5.2 DE
- Quiz { 7.1, 3, 4 e^x, $\ln x$ a^x x^x; 5.2 7.5 DE 11.5 slope field, Euler; 7.7 $\sin^{-1}$ }
- Handouts - printed out & on web. "Ch.3 notes P.7~9) & solutions

- Barron's Ch. 4 Applications of Derivative

§4.1

Anton 4.2



Anton 4.2 ex 1~3

EVT Anton 4.4 Ex 1~5

① • Does f have a max or min value? Extreme Value Thm

② • If they exist, where are they attained? Critical Points

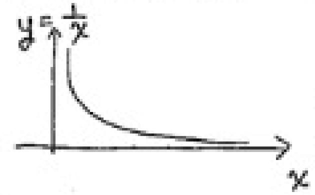
③ • What are the max, min. values?

Defn D is the domain of f. $c \in D$

- $f(c)$ is the maximum value of f on D $f(x) \le f(c)$ $\forall x \in D$
- $f(c)$ " minimum value " $f(x) \ge f(c)$ $\forall x \in D$
- $f(c)$ " an extreme value of f on D if $f(c)$ is either a max or min.
- f, the function to be maximized or minimized is the objective function

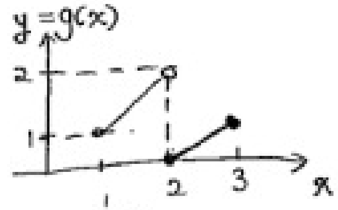
① Extreme Value Thm : f is continuous on closed interval $[a,b]$
$\Rightarrow$ f attains both a maximum & a minimum on $[a,b]$

ex



$f(x) = \frac{1}{x}$ is continuous

- On $(0,\infty)$, f has no max or min
- On $[1,3]$, $f(1) = 1$ is max
  $f(3) = \frac{1}{3}$ is min,

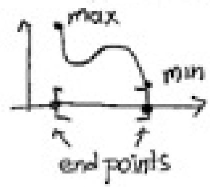


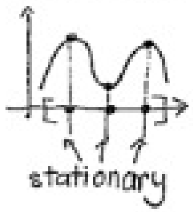
- g is defined on closed $[1,3]$ but g is not continuous
- $g(2) = 0$ is the minimum value on $[1,3]$

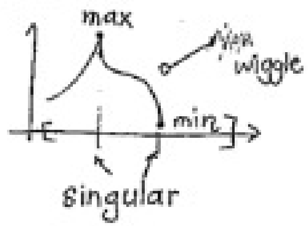
g has no maximum value. 10min

② Critical Point Thm.
f is defined on interval I. $c \in I$.
$f(c)$ is an extremum $\Rightarrow$ c is a critical point;
$\not\Leftarrow$

c is
include (1) an end point of I
(2) a stationary point of f $(f'(c) = 0)$
(3) a singular point of f $(f'(c)$ dne$)$



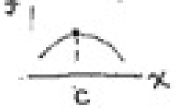




4)

Proof of Critical Point Thm

- f could attain a max or min at an endpoint or singular pt.
- Assume $f(c)$ is a maximum of f and c is <u>not</u> an endpoint or singular point f,



When $x < c$, $\dfrac{f(x)-f(c)}{x-c} \ge 0 \Rightarrow \lim_{x \to c^-} \dfrac{f(x)-f(c)}{x-c} \ge 0$

when $x > c$, $\dfrac{f(x)-f(c)}{x-c} \le 0 \Rightarrow \lim_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} \le 0$

but $f'(c)$ exists. So $f'(c) = \lim_{x\to c^-} \dfrac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \dfrac{f(x)-f(c)}{x-c}$

$= 0$

$\boxed{\text{HW}}$ Prove: $f(c)$ is a <u>minimum</u> value of f,
c is not an endpoint nor a singular point
(so $f'(c)$ exists)

$\Rightarrow f'(c) = 0$

10min

③ Finding the Absolute Max & Min Values of f

Step 1: Identify the 3 types of Critical Points
Step 2: Evaluate f at each critical point

$\boxed{\text{ex3}}$ $F(x) = x^{2/3}$ is continuous everywhere.
Find its max & minimum values on $[-1, 2]$

Solution:

1) Critical points $f'(x) = \dfrac{2}{3x^{1/3}}$

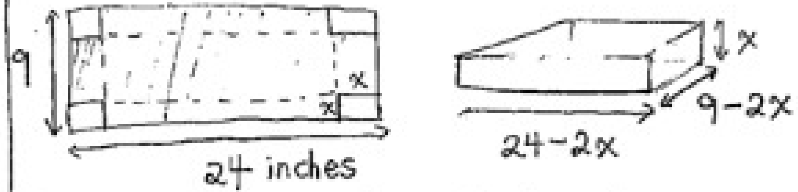
- endpoints $x = -1$, 2
- singular? $x = 0$
- Stationary? None

x	-1	0	2
$f(x)$	1	0	$\sqrt[3]{4}$

Maximum Value $\sqrt[3]{4}$ at $x = 2$
Minimum 0 at $x = 0$

10min

$\boxed{\text{ex4}}$ Rectangular box made of cardboard with sides turned up. Find the dimensions of the box that maximize the volume. What is the maximum volume?



Solution: Volume V depends on x.

$V = (24-2x)(9-2x)\,x = f(x)$ objective function
$= 216x - 66x^2 + 4x^3$

$0 \le x \le 4.5$

Critical points:

- Endpts: $x = 0, 4.5$
  $f'(x) = 216 - 132x + 12x^2 = 12(9-x)(2-x) = 0$
- Singular pts: None
- Stationary: $f'(x) = 0$ when $x = 2, \cancel{9}$

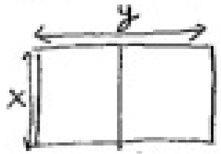
Plug:

x	0	2	4.5	$\cancel{9}$
$f(x)$	0	200	0	

Maximum is $200\ \text{in}^3$ at $x = 2$ inches
or $l \times w \times h = 20 \times 5 \times 2$ inches
dimensions

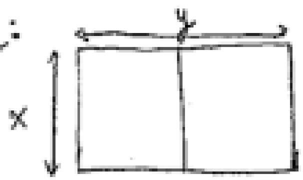
10min

$\boxed{\text{ex5}}$ A farmer has 100m of wire fencing to make 2 identical adjacent pens.
Find the dimensions to maximize the area.



10min

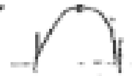
Solution:



$A = xy$

$3x + 2y = 100 \Rightarrow y = 50 - \frac{3}{2}x$

Maximize wrt. x, $A = x(50 - \frac{3}{2}x) = 50x - \frac{3}{2}x^2$



$f(x) =$

$0 \le x \le 100/3$

Critical Points:

Endpts: $0,\ 100/3$

$f'(x) = 50 - 3x$

Singular: None

Stationary: $f'(x) = 0$ when $x = \frac{50}{3}$

x	0	$50/3$	$100/3$
$f(x)$	0	416.67	0

max when

$$\boxed{\begin{aligned} x &= 50/3 \text{ m} \\ y &= 50 - \tfrac{3}{2}\left(\tfrac{50}{3}\right) \\ &= 25 \text{m} \end{aligned}}$$

$\boxed{\text{HW \#9}}$ §4.1 (P.166)

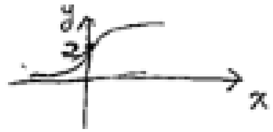
#3, 5, 9, 11, 13, 15, 17, 21, 25, 27

12/1/2009

ex6 Find all points of inflection of $F(x) = x^{1/3} + 2$

Soln: $F'(x) = \frac{1}{3} x^{-2/3}$

$F''(x) = -\frac{2}{9} x^{-5/3} \begin{cases} >0 & \text{when } x<0 \\ \text{dne} & x=0 \\ <0 & x>0 \end{cases}$



Anton 4.2 Ex 4~6, 8
Anton 4.4 Ex 6

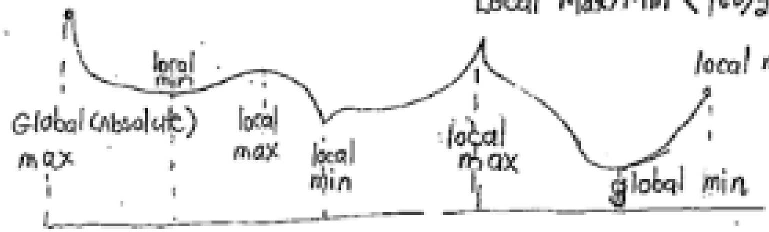
§4.3 Local Maxima & Minima

defn

- $f(c)$ is a local maximum value of f if there is an interval (a,b) containing c where $f(c) \geq f(x)$ $\forall x \in (a,b)$
- local minimum
- local extremum

* Absolute Max/Min < Critical Pts, Plug

Local Max/Min < Critical Pts, 1st/2nd derivative Test



Where do extrema occur? Critical Points

local extrema?
- Endpoint [ ]
- Singular Point $f'(x)$ dne
- Stationary Point $f'(x)=0$

Local Extremum? Global (Plug)
First Derivative Test
Second Derivative Test

Thm A First Derivative Test

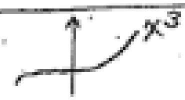
f is continuous on open interval (a,b) that contains a critical point c (singular or stationary).

(1) If $f'(x) > 0$ on (a,c) & $f'(c) < 0$ on (c,b) then $f(c)$ is a local max

(2) If $f'(x) < 0$ on (a,c) & $f'(c) > 0$ on (c,b) then $f(c)$ is a local min

(3) If $f'(x)$ has the same sign on both sides, then $f(c)$ is not a local extremum.

$f'(c)$ does not need to exist



$f'(x) = 3x^2$ does not change sign, $=0$ at 0

Proof of (i) $\begin{cases} f'(x) > 0 \text{ on } (a,c) \Rightarrow f \uparrow \text{ on } (a,c) \\ f'(x) < 0 \text{ on } (c,b) \Rightarrow f \downarrow \text{ on } (c,b) \end{cases}$ $\because$ Monotonicity Thm.

$\Rightarrow \begin{cases} f(x) < f(c) & \forall x \in (a,c) \\ f(c) > f(x) & \forall x \in (c,b) \end{cases}$ $\therefore f(x) < f(c) \quad \forall x \in (a,b)$

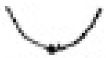
similarly for (ii) & (iii)

ex1 Local extreme values? $f(x) = x^2 - 6x + 5$ on $(-\infty, \infty)$

Soln: f continuous everywhere

$f'(x) = 2x - 6$ exists $\forall x$.

Critical Points: $f'(x) \begin{cases} = 0, & x = 3 \\ < 0 & x < 3 \\ > 0 & x > 3 \end{cases}$

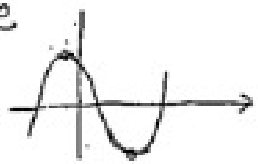


$\boxed{f(3) = -4 \text{ is a local \& global minimum}}$

ex2 Find the local extreme values of $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ on $(-\infty, \infty)$

Soln: f continuous, differentiable everywhere

Critical Points? $f'(x) = x^2 - 2x - 3$
$= (x-3)(x+1)$



$x = 3, -1$

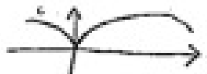
x		-1		3	
$f'(x)$	$+$	0	$-$	0	$+$
	↗		↘		↗

$\boxed{\begin{array}{l} f(-1) = \frac{17}{3} \text{ local max} \\ f(3) = -5 \text{ local min} \end{array}}$

ex3 Find the local extreme values of $f(x) = (\sin x)^{2/3}$ on $(-\frac{\pi}{6}, \frac{2\pi}{3})$

Soln: $\because f'(x) = \dfrac{2\cos x}{3(\sin x)^{1/3}}, \quad x \neq 0$

Critical Points $\begin{cases} = 0 & \text{when } x = \pi/2 \\ \text{dne} & \text{when } x = 0 \end{cases}$



x		0		$\pi/2$	
$f'(x)$	$\frac{+}{-}$ ⊖	0	$\frac{+}{+}$ ⊕	dne	$\frac{-}{+}$ ⊖
	↘		↗		↘

$\Rightarrow \boxed{\begin{array}{l} f(0) = 0 \text{ local min} \\ f(\frac{\pi}{2}) = 1 \text{ local max} \end{array}}$

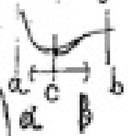
Thm B Second Derivative Test

f'' (and f') exist at every point in open interval (a,b) containing c and $f'(c) = 0$

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum value of f

(ii) If $f''(c) > 0$, then $f(c)$ is a local minimum val. of f

Proof (i) It's not enough to say f is concave down so $f(c)$ is a maximum, since $f''(c) < 0$ does not guarantee $f''(x) < 0$ $\forall x$ in some open neighborhood.



$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x) - 0}{x - c} < 0$$

$\Rightarrow$ There is some interval (α, β) around c where

$$\frac{f'(x)}{x - c} < 0, \quad x \neq c$$

$\Rightarrow f'(x) \begin{cases} < 0 & \text{when } \alpha < x < c \\ > 0 & \text{when } c < x < \beta \end{cases}$

HW (ii).

ex4 Local extremes? $f(x) = x^2 - 6x + 5$. (see ex1)

$f''(x) = 2 > 0$

$f'(3) = 0$, $f''(3) > 0$ $\therefore f(3)$ is local max

ex5 $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. Local extremes?

$\begin{cases} f'(x) = (x-3)(x+1) \\ f''(x) = 2x - 2 \end{cases}$

Critical Points: $-1, 3$

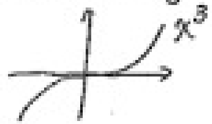
$\because f'(-1) = f'(3) = 0$

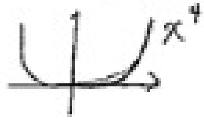
$f''(-1) = -4 < 0 \Rightarrow$ Local max $f(-1)$

$f''(3) = 4 > 0 \Rightarrow$ local min $f(3)$

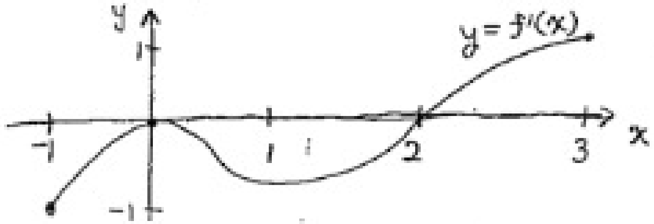
When the 2nd Derivative Test fails:

$f''(x)$ may be 0 at a stationary point



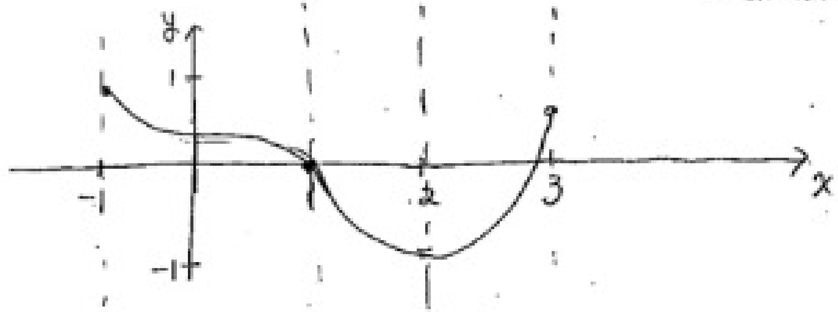


ex6 A plot of $y=f'(x)$ is shown. Find all local extrema, and points of inflection of f on the interval $[-1,3]$. Given $f(1)=0$, sketch the graph of $y=f(x)$?



- Sketch the graph of $y=f(x)$
- Make a Table with x = critical points: singular: none; endpt $-1, 3$; stationary: $0, 2$
  & where $f(x)=0$, x-intercept $(1,0)$, $f(0)$ = y-intercept dunno

x	-1		0		1		2		3
$f'(x)$		$-$	0	$-$		$-$	0	$+$	
$f''(x)$		$+$	0	$-$	0	$+$		$+$	
$f(x)$	local max		i		0 i		local & absolute minimum		local max



Anton 4.2 Polynomial Multiplicity

Anton 4.3 Ex1~4

§4.6 Sophisticated Graphing
Anton 4.3

Method

Step 1: Precalculus
- Domain, range
- Symmetry (odd? even?)
- Intercepts

Step 2: Calculus
- $f'(x) \Rightarrow$ critical points $\uparrow \downarrow$?
- local max? min?
- $f''(x)$ $\cup$ $\cap$
- Asymptotes

Step 3: Plot all critical & inflection points

Step 4: connect the dots

① Domain, Range, Asymptotes? (Horiz. Vertical Slant)
② Table x, $f'(x)$, $f''(x)$, $f(x)$ — Critical pts, where $f'(x)=0$, dne; $f''(x)=0$
③ Sketch

Polynomial

ex1 Sketch $f(x) = \dfrac{3x^5 - 20x^3}{32}$. Label all local extrema & x-intercepts & inflection points

Soln: $f'(x) = \dfrac{15x^2(x-2)(x+2)}{32}$

① Domain $x \in \mathbb{R}$ Asymp: none
② Critical points? Endpt: none
singular: none
stationary: $f'(x)=0$ at $x=0,-2,2$
$f(x)$ exists everywhere

$$f''(x) = \frac{15x(x-\sqrt{2})(x+\sqrt{2})}{8} = 0 \text{ when } x = 0, \sqrt{2}, -\sqrt{2}$$

③

x	$-\infty$		-2		$-\sqrt{2}$		0		$\sqrt{2}$		2		∞
$f'(x)$		$+$	0	$-$		$-$	0	$-$		$-$	0	$+$	
$f''(x)$		$-$		$-$	0	$+$	0	$-$	0	$+$	0	$+$	
$f(x)$			2 local		1.2		0		-1.2		-2 local		

Rational

Label local max, min

$\boxed{ex2}$ $f(x) = \dfrac{x^2-2x+4}{x-2}$. Sketch.

Soln: ① Domain? $x \neq 2$

Asymptotes? $y = \dfrac{x^2-2x+4}{x-2} = \dfrac{(x-2)x+4}{x-2} = x + \dfrac{4}{x-2}$

$\begin{cases} \text{vertical: } x=2 \\ \text{horiz: none} \\ \text{slant: (num 1 deg bigger) } \boxed{y=x} \end{cases}$

② Table Critical Pts: Endpt: None

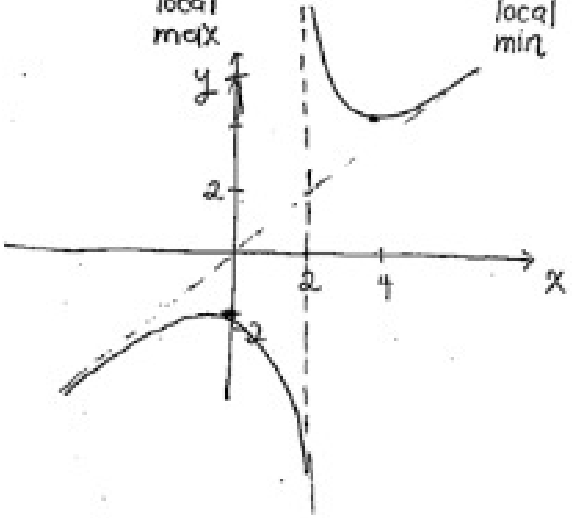
$f'(x) = \dfrac{x(x-4)}{(x-2)^2}$ Singular: $x=2$

Stationary: $x=0,4$

$f''(x) = \dfrac{8}{(x-2)^3} \begin{cases} >0 & x>2 \\ <0 & x<2 \end{cases}$

x	$-\infty$		0		2		4		$+\infty$
$f'(x)$		+	0	−	DNE	−	0	+	
$f''(x)$		−		−	DNE	+		+	
$f(x)$			−2		DNE		6		
			local max				local min		

③



Functions With Roots

$\boxed{ex3}$

Functions with Roots (§4.6 ~~Sophisticated~~ Graphing)

ex3 Sketch $F(x) = \dfrac{\sqrt{x}\,(x-5)^2}{4}$
HW

Soln: 1. Domain: $x \ge 0$

Asymptotes: None

2. Table $F'(x) = \left[\dfrac{(x-5)^2}{2\sqrt{x}} + \sqrt{x}\ 2(x-5)\right]\dfrac{1}{4}$

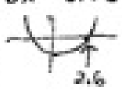
$= \dfrac{(x-5)^2 + 4x(x-5)}{8\sqrt{x}} = \dfrac{(x-5)(x-5+4x)}{8\sqrt{x}}$

$= \dfrac{5(x-5)(x-1)}{8\sqrt{x}}$

$F''(x) = \dfrac{5(3x^2-6x-5)}{16x^{3/2}}$

Critical Points: Endpt: $x=0$
Stationary: $x = 1, 5$
Singular: $x=0$

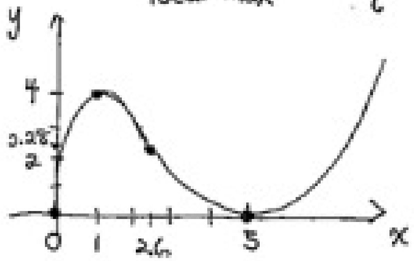
$3x^2-6x-5$



Possible inflection pt
$F''(x)=0$ at $3x^2-6x-5=0$
$x = 1 + \dfrac{2\sqrt{6}}{3} \in [0,\infty)$
≈ 2.6
$F''(x)$ dne at $x=0$

x	0		1		2.6		5		$+\infty$
$f'(x)$	DNE	+	0	−		−	0	+	
$f''(x)$	dne	−		−	0	+		+	
$f(x)$	0	⌒	4 local max	⌒	2.28 i	⌣	0 local min	⌣	

* Vertical Tangent



Anton 4.5
Ex 1~6
economics Ex7 *

§4.4 More Max-Min Problems

The domain may be closed, open, half-open / half-closed
[ ] ( ) [ ) ( ]

Apply the correct theory to find (global) max & min.

1. Get Objective function in terms of 1 variable
2. Critical Points
   - Endpt
   - Singular — First Deriv Test → local max/min
   - Stationary — Second Deriv. Test → Maybe global
3. Use First or Second Deriv. Test &/or plug in points for max/min.

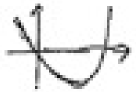
Extrema on Open Intervals

ex1 Find (if any exist) the maximum and minimum values of $f(x) = x^4 - 4x$ on $(-\infty, \infty)$

Soln: Critical Pts
End: none
Singular: none
Stationary: $x=1$

$f'(x) = 4x^3 - 4 = 4(x-1)\underbrace{(x^2+x+1)}$
$1^2 - 4(1)(1) < 0$
no zero

$f'(x) \begin{cases} <0 & x<1 \\ =0 \text{ at } & x=1 \\ >0 & x>1 \end{cases}$



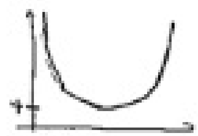
∴ $\boxed{f(1) = -3 \text{ is a local \& global minimum,}}$ since f decreases on the left of 1 and increases on the right of 1.

ex2 Find (if any exist) the maximum & minimum values of $G(p) = \dfrac{1}{p(1-p)}$ on $(0,1)$

Solution: $G'(p) = -\dfrac{1}{p^2(1-p)} + \dfrac{1}{p(1-p)^2} = \dfrac{-(1-p)+p}{p^2(1-p)^2} = \dfrac{2p-1}{p^2(1-p)^2}$

Critical Pt: End: None
Singular: ~~$p=0,1$~~ $\notin (0,1)$
Stationary: $p = 1/2$

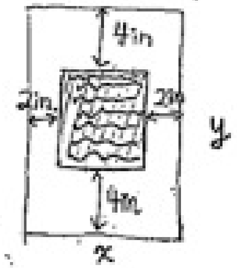
$G'(p) \begin{cases} <0 & p<1/2 \\ =0 & p=\frac{1}{2} \\ >0 & >1/2 \end{cases}$



G decreases on $(0, 1/2)$
increases on $(1/2, 1)$
By the 1st Deriv. Test
$\boxed{G(1/2) = 4 \text{ is a global min}}$

Practical Problems (§4.4)

ex3 A handbill is to contain 50 square inches of printed matter, with 4-inch margins at top and bottom and 2-inch margins on each side. What dimensions for the handbill would use the least paper?



Solution: Guess $x<y$ to capitalize on narrow margins along the side.

1. Minimize Area. Get in terms of 1 variable.

$A = xy \qquad (x-4)(y-8)=50$

$y = \frac{50}{x-4}+8$

$\Rightarrow \boxed{A = \frac{50x}{x-4}+8x}$

2. Critical Points

$x \in (4,\infty)$

endpt: none

Singular: $\cancel{x=4}$

Stationary: $x = \cancel{-1}, 9$

$\notin (4,\infty)$

$A'(x) = \frac{(x-4)50-50x}{(x-4)^2}+8$

$= \frac{8x^2-64x-72}{(x-4)^2} = \frac{8(x+1)(x-9)}{(x-4)^2}$

3. $\begin{cases} <0 & x<9 \\ =0 & x=9 \\ >0 & x>9 \end{cases}$

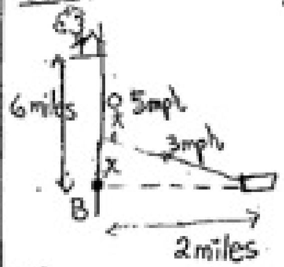
$x=9$ gives a <u>global</u> min for area

$y = \frac{50}{9-4}+8 = 18$

$\boxed{9\text{ inches} \times 18\text{ inches}}$

ex4 Andy, who is in a rowboat 2 miles from the nearest point B on a straight shoreline, notices smoke billowing from his house, which is 6 miles down the shoreline from B. He figures he can row at 3 miles per hour and run at 5 miles per hour. How should he proceed in order to get to his house in the least amount of time?

Soln:



1. Minimize time. Get time in terms of x.

$T = t_{row} + t_{run}$

$3t_{row} = \sqrt{x^2+2^2}$

$5t_{run} = 6-x$

$\Rightarrow \boxed{T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5} \quad x\in[0,6]}$

2. Critical Points

End: $x=0,6$

Singular: None

Stationary:

$5x = 3\sqrt{x^2+4}$

$25x^2 = 9(x^2+4)$

$16x^2 = 36$

$x = \pm\frac{3}{2}$

$x = \frac{3}{2}$

$T'(x) = \frac{1}{3}\cdot\frac{1}{2}\frac{2x}{\sqrt{x^2+4}} - \frac{1}{5}$

$= \frac{5x-3\sqrt{x^2+4}}{15\sqrt{x^2+4}}$

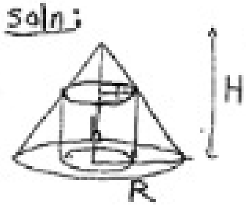
x	T
0	1.87
1.5	1.73
6	2.11

(1st Deriv. Test hard)

Answer: Andy should row to $\boxed{x=1.5\text{ miles}}$ then run. This would take a minimum time of $\boxed{1.73\text{ hrs}}$

ex5 Find the dimensions of the right circular cylinder of greatest volume that can be inscribed in a given right circular cone.

Soln:



Given R, H of cone

Find r, h of cylinder

1. Maximize $V_{cylinder} = \pi r^2 h$

Easier to get h in terms of r

Get in 1 variable $\frac{H}{R} = \frac{h}{R-r}$

$h = \frac{H}{R}(R-r) = H - \frac{H}{R}r$

$\boxed{\begin{aligned} V(r) &= \pi r^2\left(H-\frac{H}{R}r\right) \\ &= \pi H r^2 - \frac{H}{R}\pi r^3 \end{aligned}}$

$\Rightarrow V'(r) = 2\pi H r - \frac{3\pi H}{R}r^2 = \pi H r\left(2-\frac{3}{R}r\right)$

2. Critical Points: $r \in [0, R]$

endpt: $r = 0, R$

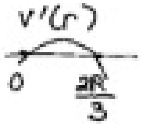
Singular: None

Stationary: $r=0, \frac{2R}{3}$

$V(0)=V(R)=0$

$\therefore V(\frac{2R}{3})$ max

$$\boxed{r=\frac{2R}{3}\ ,\ h=H-\frac{H}{R}\left(\frac{2R}{3}\right)=\frac{H}{3}}$$



r	0		$2R/3$		R
$v'(r)$	0	$+$	0	$-$	
$v(r)$	0		global max		0

Linear Least-Squares Regression

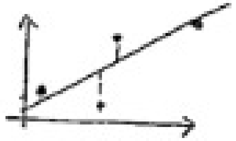
Newton's Second Law $F=ma$

Hooke's Law $F=kx$

F	a

x	F

Scatter Plot



Find the best-fit line $y=mx+b$, near points $(x_1,y_1),(x_2,y_2)$ $,\dots,(x_n,y_n)$

Find (m,b) that minimize the sum of squared distances from each point to the line. $y=mx+b$ is the least-squares line

Goal: Minimize $\sum_{i=1}^{n}[y_i-(mx_i+b)]^2=F(m,b)$

② $\hat{m}=\arg\min_m F(m,b)$ ← view b as constant

① $\hat{b}=\arg\min_b F(m,b)$ ← View m as constant

$(\hat{m},\hat{b})$ will minimize $F(m,b)$

① $F(b)=\sum_{i=1}^{n}[y_i-mx_i-b]^2$

Critical Points: Endpt-none. Singular: None. Stationary?

$$\frac{\partial F(m,b)}{\partial b}=\frac{dF(b)}{db}=-\sum_{i=1}^{n}2(y_i-mx_i-b)\overset{?}{=}0$$

$$\sum_{i=1}^{n}(y_i-mx_i-b)=0$$

$$\sum_{i=1}^{n}y_i-m\sum_{i=1}^{n}x_i=b\sum_{i=1}^{n}1$$

$$\boxed{\hat{b}=\frac{\sum_{i=1}^{n}y_i-\hat{m}\sum_{i=1}^{n}x_i}{n}}$$

$$\frac{\partial^2F(m,b)}{\partial b^2}=\sum_{i=1}^{n}1=n>0\quad\forall b\in\mathbb{R}$$ always concave up (not just at $\hat{b}$)

$$\therefore\ \hat{b}=\arg\min_b F(m,b)$$

② $$\frac{\partial F(m,b)}{\partial m}=\frac{dF(m)}{dm}\qquad F(m)=\sum_{i=1}^{n}[y_i-(mx_i+b)]^2$$

$$=\sum_{i=1}^{n}\{2[y_i-(mx_i+b)](-x_i)\}$$

Critical Pts: end-none, singular-none, stationary?

$$\sum_{i=1}^{n}x_iy_i-m\sum_{i=1}^{n}x_i^2-b\sum_{i=1}^{n}x_i=0$$

$$\hat{m}\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}x_iy_i-\hat{b}\sum_{i=1}^{n}x_i$$

$$\hat{m}\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}y_i\sum_{j=1}^{n}x_j+\frac{1}{n}\hat{m}\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j$$

$$\hat{m}\left[\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2\right]=\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)$$

$$\boxed{\hat{m}=\frac{\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)}{\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2}}$$

Next: MVT (Anton 4.7 VPR 4.7)

Anton 4.8 Rectilinear Motion

12/1/2009

✓ MVT
✓ Monotonicity, Concavity
✓ Graphing by hand

§4.7 The Mean Value Thm

Anton 4.7 Ex1~5

There is at least one point $C \in (A, B)$ where the tangent line is parallel to the secant line joining A & B

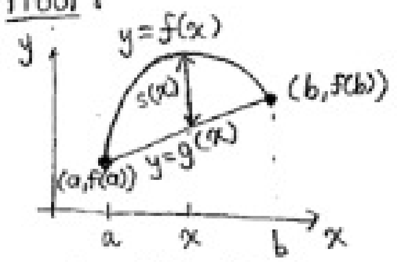
☆ Thm. **The Mean Value Thm for Derivatives**

f is differentiable on open interval (a,b) & continuous on $[a,b]$

$\Rightarrow \exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof:



$s(a) = s(b) = 0$

Idea: Show $\exists\, x \in (a,b)$ so $\underbrace{f'(x)}_{\text{slope of } f} - \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of line } g(x)}$

Let $s(x) = f(x) - g(x)$

$s'(x) = f'(x) - g'(x)$

$\quad = f'(x) - \frac{f(b)-f(a)}{b-a}$.

- Where is $s'(x) = 0$? (Max/min)
- s is continuous because f & g are continuous. By the Extreme Value Theorem, $s(x)$ must have a maximum and a minimum on $[a,b]$.
- If max = min, $s(x) = 0$ on $[a,b]$, so $s'(x) = 0$, $f'(x) = \frac{f(b)-f(a)}{b-a}$ $\forall x \in (a,b)$
- If max ≠ min, then one of them is not 0, so the extremum is achieved at some $x \in (a,b)$ $x \neq a, b$. By the Critical Point Thm, $s'(x) = 0$. So $\exists\, x \in (a,b)$ s.t. $s'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$.

☆ **Extreme Value Thm**

f is continuous on closed interval $[a,b]$

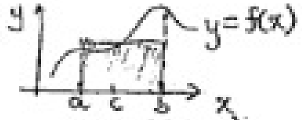
$\Rightarrow$ f attains both a maximum value and a minimum value.

☆ **Intermediate Value Thm**

f is continuous on $[a,b]$.

$f(a) \le M \le f(b) \Rightarrow \exists\, c \in [a,b]$ s.t. $f(c) = M$

☆ **Mean Value Thm for Integrals**

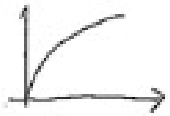


f continuous on $[a,b]$

$\Rightarrow \exists\, c \in [a,b]$ s.t. $\int_a^b f(x)\,dx = f(c)(b-a)$

ex1 $f(x) = 2\sqrt{x}$ on $[1,4]$

Find the value c guaranteed by MVT



Soln: $f'(c) = \frac{1}{\sqrt{c}} = \frac{f(4)-f(1)}{4-1} = \frac{4-2}{3} = \frac{2}{3} \Rightarrow \boxed{c = \frac{9}{4}}$

ex2 $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$. Find all numbers c that satisfy the conclusion of MVT.

Soln:

$f'(x) = 3x^2 - 2x - 1$ $\qquad \frac{f(2)-f(-1)}{2--1} = 1$

$3c^2 - 2c - 1 = 1$

$3c^2 - 2c - 2 = 0 \Rightarrow c = \frac{2 \pm \sqrt{4+24}}{6} = -0.55,\ \underline{1.22}$

ex3 $f(x) = x^{2/3}$ on $[-8, 27]$. Show the conclusion of MVT fails. Why?

Solution: $f'(x) = \frac{2}{3}x^{-1/3}$, $\underline{x \neq 0}$ not differentiable at $x=0$.

problem: f not differentiable everywhere on (a,b)

$\frac{f(27)-f(-8)}{27-(-8)} = \frac{1}{7}$

$\frac{2}{3}c^{-1/3} = \frac{1}{7}$

$c = \left(\frac{14}{3}\right)^3 \approx 102 \notin (-8, 27)$

Using MVT to prove other theorems

★ **Thm.** $F'(x) = G'(x) \ \forall x \in (a,b)$

$\Rightarrow \exists\, c \in \mathbb{R}$ s.t. $F(x) = G(x) + C \ \forall x \in (a,b)$

Proof: Let $H(x) = F(x) - G(x)$. Show $H(x) = H(x_1)$ constant

Let $x \in (a,b)$. Let $x_1 \in (a,b)$

H is continuous on $[x, x_1]$ and differentiable on (x, x_1)

By MVT, $\exists\, c \in (x, x_1)$ s.t.

$H(x) - H(x_1) = \underbrace{H'(c)}(x - x_1)$

$\therefore H(x) = H(x_1)$ but $H'(x) = F'(x) - G'(x) = 0$ always

---

§4.2 Monotonicity and Concavity (Anton 4.1)

increasing: $f(x_1) \le f(x_2) \iff x_1 \le x_2$

strictly: $f(x_1) < f(x_2) \iff x_1 < x_2$

monotonic: strictly increasing or strictly decreasing

> **Thm. A** Monotonicity Thm.
> f continuous on interval I and differentiable in I
> - $f'(x) > 0 \ \forall x \in I \Rightarrow f$ strictly increasing on I
> - $f'(x) < 0 \ \forall x \in I \Rightarrow f$ strictly decreasing on I

Proof by MVT

$f'(x) > 0 \ \forall x \in I = [a,b]$

Let $x_1 < x_2 \in (a,b)$ (show $f(x_2) - f(x_1) > 0$)

$\exists\, c \in (x_1, x_2)$ s.t.

$\underbrace{f(x_2) - f(x_1)} = \underbrace{f'(c)}_{>0}\underbrace{(x_2 - x_1)}_{>0}$

$\therefore f(x_2) > f(x_1)$

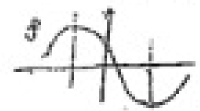
(HW) Prove the case when $f'(x) < 0$.

---

ex1 $f(x) = 2x^3 - 3x^2 - 12x + 7$.
Where is f increasing? decreasing?

Soln: $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$

f increasing on $(-\infty, -1] \cup [2, \infty)$
decreasing on $[-1, 2]$

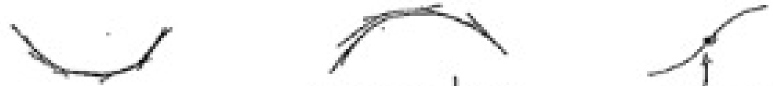


ex2 Where is $g(x) = \dfrac{x}{1+x^2}$ increasing? decreasing?

Soln: $g'(x) = \dfrac{(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2} = \dfrac{(1-x)(1+x)}{(1-x^2)^2} = 0$, $x = 1, -1$

$g(x)$ increasing on $[-1, 1]$
decreasing on $(-\infty, -1] \cup [1, \infty)$

Definition



concave up / convex — f' increasing on open interval I

concave down / concave — f' decreasing on open interval

inflection point

> **Thm B** Concavity Theorem
> $f''(x) > 0 \ \forall x \in I \Rightarrow f$ concave up on I
> $f''(x) < 0 \ \forall x \in I \Rightarrow f$ concave down on I

Proof: $f''(x) > 0 \Rightarrow f'$ increasing on I, so concave up

ex4 Where is $g(x) = \dfrac{x}{1+x^2}$ concave up? down? sketch.

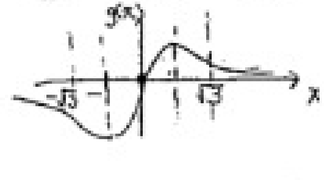
Soln: $g''(x) = \dfrac{(1+x^2)^2(-2x) - (1-x^2)(2)(1+x^2)(2x)}{(1+x^2)^4}$

$= \dfrac{(1+x^2)[(1+x^2)(-2x) - (1-x^2)(4x)]}{(1+x^2)^4}$

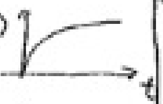
$= \dfrac{2x^3 - 6x}{(1+x^2)^3} = \dfrac{2x(x^2-3)}{(1+x^2)^3} = 0$ at $x = 0, \sqrt{3}, -\sqrt{3}$

Table: critical points, endpts, $f'(x)=0$ dne, $f''(x)=0$

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
g'	−		−	0	+		+	0	−		−
g''	−	0	+		+	0	−		−	0	+
g						0					

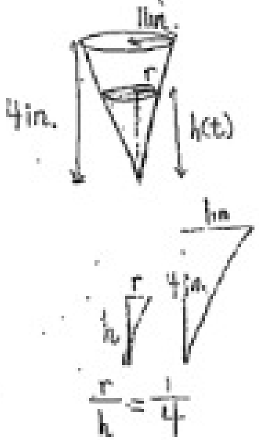


ex5 Water is poured into a conical container at a constant rate of $\frac{1}{2}\,in^3$ per second.
Find height as function of time t and plot $h(t)$ from time $t=0$ until the time the container is full.



Soln: Idea - h increases faster, then slower
$h'(t)>0$ increase
$h''(t)<0$ (slope gets smaller) concave down

Question: Can a function always be increasing & concave down?



$\frac{r}{h}=\frac{1}{4}$

$\frac{dv}{dt}=\frac{1}{2}\,in^3/sec$

fn. of t

$$\begin{cases} V(h(t)) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h = \frac{\pi}{48}h^3 \\ V = \frac{1}{2}t + \cancel{C} \qquad V(0)=0=C \end{cases}$$

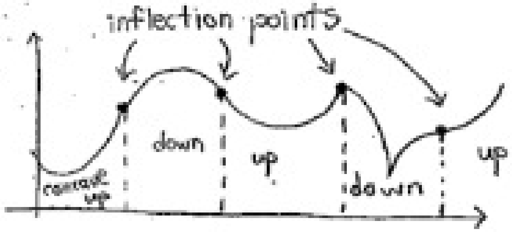
$$\frac{\pi}{48}h^3 = \frac{1}{2}t \Rightarrow \boxed{h(t) = \sqrt[3]{\frac{24}{\pi}t}}$$

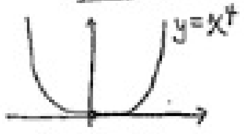
Graph:

$$h'(t) = \frac{1}{3}\left(\frac{24}{\pi}t\right)^{-2/3}\left(\frac{24}{\pi}\right) = \frac{8}{\pi}\left(\frac{\pi}{24t}\right)^{2/3}$$
$$= \frac{2^3}{(2^3 3)^{2/3}\pi^{1/3}t^{2/3}} = \frac{2}{\sqrt[3]{9\pi t^2}} > 0$$

$$h''(t) = 2 \times -\frac{1}{3}\,(9\pi t^2)^{-4/3} \times 2t = -\frac{4}{3\sqrt[3]{9\pi t^5}} < 0$$

Inflection Point $(c, f(c))$ f is concave up on one side & concave down on the other.
$\boxed{\begin{array}{l}\Rightarrow f''(c)=0 \text{ or dne} \\ \not\Leftarrow\end{array}}$





$f'(x)=4x^3$
$f''(x)=12x^2$

$f''(0)=0$ but NOT inflection point

ex6 Find all points of inflection of $F(x)=x^{1/3}+2$

Soln: $F'(x)=\frac{1}{3}x^{-2/3}$
$F''(x)=-\frac{2}{9}x^{-5/3}\begin{cases} >0 & \text{when } x<0 \\ \text{dne} & x=0 \\ <0 & x>0\end{cases}$

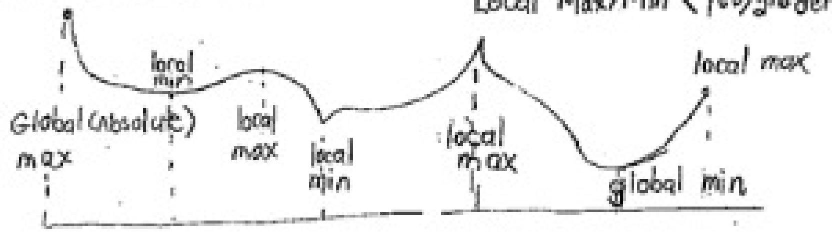
Anton 4.2 Ex 4~6, 8
Anton 4.4 Ex 6

§4.3 Local Maxima & Minima

defn
- $f(c)$ is a local maximum value of f if there is an interval (a,b) containing c where $f(c) \geq f(x)$ $\forall x \in (a,b)$
- local minimum
- local extremum

Absolute Max/Min $<$ Critical Pts, Plug

★ Local Max/Min $<$ Critical Pts, 1st/2nd derivative Test

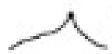


Where do extrema occur? Critical Points

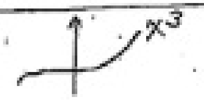
local extrema?
- Endpoint [ ]
- Singular Point $f'(x)$ dne
- Stationary Point $f'(x)=0$

Global (Plug)
Local Extremum? First Derivative Test
Second Derivative Test

Thm A First Derivative Test
f is continuous on open interval (a,b) that contains a critical point c (singular or stationary)
(1) If $f'(x)>0$ on (a,c) & $f'(c)<0$ on (c,b) then $f(c)$ is a local max
(2) If $f'(x)<0$ on (a,c) & $f'(c)>0$ on (c,b) then $f(c)$ is a local min
(3) If $f'(x)$ has the same sign on both sides, then $f(c)$ is not a local extremum.



$f'(c)$ does not need to exist



$f'(x)=3x^2$ does not change sign
$=0$ at 0