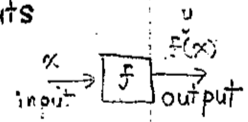
Functions

- Special Functions II P.9 III P.9 V P.11
  - Polynomial (linear, quadratic, higher-order)
  - VI P.14 – Rational VII P.16 Limits IV P.10 Shortcuts
  - – Absolute Value
  - P.17 VIII – Piecewise defined IX P.18 Parametric
  - I Pg7 – Exponential, Logarithmic
  - – Trigonometric



I Definitions

Relation = set of ordered pairs

$\{(0,10), (0.1, 9.8), (0.2, 9.4)\}$
(x, y)

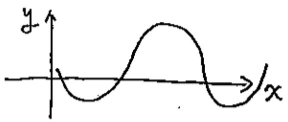
Function: a relation where each first element is paired with a unique second element
(Each x-value is mapped to only one y-value)

$\boxed{\text{ex4}}$ a) $\{(-2,-1), (-2,3), (0,3), (5,4)\}$ Function? No

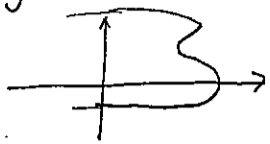
- Mapping diagram

$-2 \longrightarrow -1$
$-2 \longrightarrow 3$
$0 \longrightarrow 3$
$5 \longrightarrow 4$

$\boxed{\text{ex}}$ **Vertical Line Test**: Is y a function of x?



yes



no

If the vertical line passes through two or more points on the graph, then y is not a function of x

$\boxed{\text{ex}}$ $\{(x,y) : y^2 = x\}$ Is this relation a function?

No $y = \pm\sqrt{x}$ $x = 4 \to 2,\ -2$

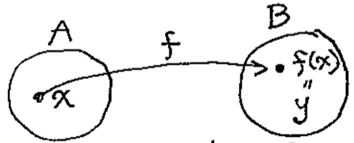
Function as a mapping. The value x is mapped to another value called "$f(x)$" "f of x", or "y"

$f: A \longrightarrow B$
$x \longmapsto f(x) = y$

x	y
independent variable	dependent variable
input	output
abscissa	ordinate

Domain: A **Codomain**: B

Range: $\{f(x) : x \in A\} \subset B$

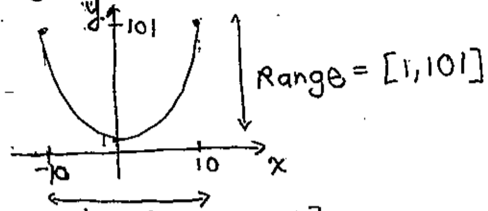


x is the pre-image or "pullback" of y | $y = f(x)$ is the image of x

- f maps the domain A into the codomain B
  f maps A onto B if the range of f is B

$\boxed{\text{ex}}$ $f: [-10, 10] \longrightarrow \mathbb{R}$
$x \longmapsto x^2+1$

$y = f(x) = x^2+1$

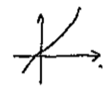


domain = [-10, 10]

Codomain: $\mathbb{R}$

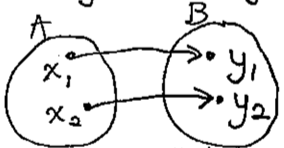
- $f(2) = 5$
- Image of 2 is 5
- Pre-image/pullback of 5 is 2

We usually deal with functions that map real numbers to real numbers.
$f: \mathbb{R} \longrightarrow \mathbb{R}$

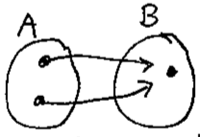
- A function is increasing if $x \le u \Leftrightarrow f(x) \le f(u)$ 
- A function is decreasing if $x \le u \Leftrightarrow f(x) \ge f(u)$ 
- A function is one-to-one if $x \neq y \Rightarrow f(x) \neq f(y)$
  which is the same as: $f(x) = f(y) \Rightarrow x = y$

(II.) Function Inverses

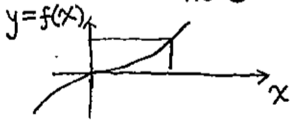
Defn $f: A \to B$, $x \mapsto y$ is <u>invertible</u> if you can pull each point y in the range back to just <u>one point</u> in the domain.
(Each y has only one x)

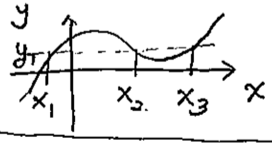


Invertible

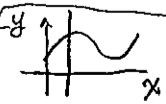


Not Invertible



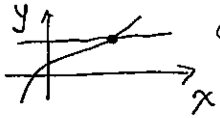


Vertical Line Test



crosses curve in at most 1 point means y is a function of x

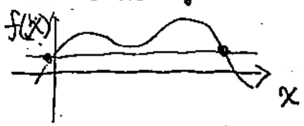
Horizontal Line Test



crosses curve in at most 1 point means y is an invertible function of x

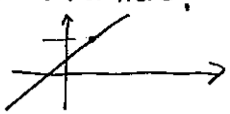
★ A function is invertible $\iff$ A function is <u>one-to-one</u> and <u>onto</u>

ex a) Invertible?

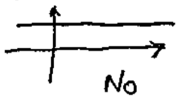


No

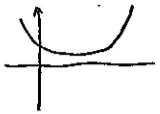
b) Invertible? yes.



c) Invertible?



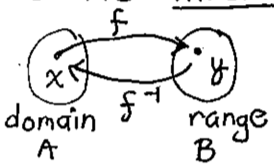
d) Function? Invertible?



e) Function? Invertible? $\{(1,2),(3,4),(5,6),(6,1),(2,2)\}$
Fn, Each x has only one y,
Not Invertible. The pre-image of 2 is 1 and 2.

If f is invertible, the pullback, f^{-1} is called the <u>inverse</u> mapping of f.

①

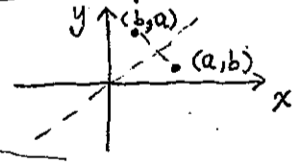


$f^{-1}: B \longrightarrow A$

$\forall y \in B,\ \underbrace{(f\circ f^{-1})(y)}_{f(f^{-1}(y))} = y$

$\forall x \in A,\ \underbrace{(f^{-1}\circ f)(x)}_{f^{-1}(f(x))} = x$

② The graph of the inverse is the reflection across the $y=x$ line. Graph f has point (a,b); Graph f^{-1} has point (b,a)

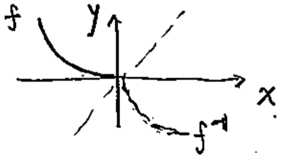


③ How to find the inverse:
1. $y=f(x)$
2. Get x in terms of y
3. Switch x and y

ex $f(x)=x^2$ with domain $x \le 0$. Find the Inverse

1. $y=x^2$
2. $x=\pm\sqrt{y}$ but since $x\le 0$, we have
   $x=-\sqrt{y}$
3. $y=-\sqrt{x}$

$\boxed{f^{-1}(x)=-\sqrt{x}}$



ex $f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $x>0$

$f^{-1}(0.33)=?$

Soln: Solve for x $\quad \frac{1}{\sqrt{2\pi}}e^{-x^2/2}=0.33$

$x^2=-2\ln(0.33\sqrt{2\pi})$

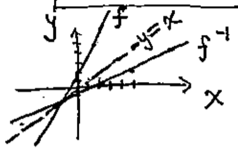
$x=\oplus\sqrt{-2\ln(0.33\sqrt{2\pi})}=0.616$

ex Find the inverse of $f(x)=3x+2$

Soln: $y=3x+2$

$x=\frac{y-2}{3}$

$f^{-1}(x)=\frac{x-2}{3}$



Check: $f\circ f^{-1}(y)=f(f^{-1}(y))=f\left(\frac{y-2}{3}\right)=3\left(\frac{y-2}{3}\right)+2=y$

$f^{-1}\circ f(x)=f^{-1}(f(x))=f^{-1}(3x+2)=\frac{(3x+2)-2}{3}=x$

(III) Domain and Range of Functions

For a function that maps a real number to a real number, $f: \mathbb{R} \longrightarrow \mathbb{R}$

<u>Domain</u>: The set of all numbers x where $f(x)$ is defined and real

- Exclude where the denominator is 0
- Exclude where there's a negative under $\sqrt{\ }$; $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, $\sqrt[n]{\ }$ (n even integer)

<u>Range</u> The set of all values mapped from the domain

- $x^n \geq 0$ n even integer
- $\sqrt[n]{x} \geq 0$, n even integer
- $|x| \geq 0$
- $-1 \leq \sin x \leq 1$
  $-1 \leq \cos x \leq 1$

ex What is the domain of $f(x) = \dfrac{x+2}{x-2}$?

<u>Soln</u>: $\{x \in \mathbb{R} : x \neq 2\}$

ex What is the domain of $g(x) = \sqrt{(x+5)(x-1)}$?

<u>Soln</u>: $(x+5)(x-1) \geq 0$

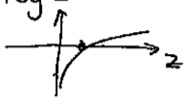
$x+5 \geq 0$ & $x-1 \geq 0$ OR $x+5 \leq 0$ & $x-1 \leq 0$

$\underbrace{x \geq -5 \ \& \ x \geq 1}_{x \geq 1}$ $\underbrace{x \leq -5 \ \& \ x \leq 1}_{x \leq -5}$

Domain: $\boxed{[1, \infty) \cup (-\infty, -5]}$

ex What is the domain of $f(x) = \log(\sqrt{x^2-1})$?

<u>Soln</u>: $\log z$



$x^2 - 1 > 0$

$\sqrt{x^2} > \sqrt{1}$

$|x| > 1$

$x > 1$ or $x < -1$

$\boxed{(-\infty, -1) \cup (1, \infty)}$

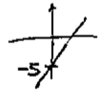
ex What is the range? $f(x) = |-x^2-8|$

Soln: $|-x^2-8| = |\underbrace{x^2+8}_{\geq 0}| = x^2 + 8$

$\boxed{(8, \infty)}$

ex What is the range of $f(x) = 4x - 5$, $0 \leq x \leq 10$

Soln:



f is an increasing function, so it's range is from $f(0)$ increasing to $f(10)$

$[f(0), f(10)] = \boxed{[-5, 35]}$

ex Find the range of each function:

	Answer:		
(1) $-x^4$	$(-\infty, 0]$	(4) $\sqrt[3]{x}$	Range: $\mathbb{R}$
(2) $-\sqrt{x}$	$(-\infty, 0]$	(5) $x^5 + 100$	$\mathbb{R}$
(3) $\lvert x\rvert - 2$	$[-2, \infty)$	(6) $\dfrac{-x^2+6}{2}$	$(-\infty, 3]$

(IV) Composition and Algebra of Functions

Definitions of function...

addition $(f+g)(x) = f(x) + g(x)$

subtraction $(f-g)(x) = f(x) - g(x)$

multiplication $(fg)(x) = f(x)\,g(x)$

division $(f \div g)(x) = \dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ (where $g(x) \neq 0$)

Composition $f \circ g(x) = f(g(x))$ (Note $f \circ g \neq g \circ f$ in general)

"f of g of x"

ex $f(x) = 3x - 2$

$g(x) = x^2 - 4$

$\dfrac{f}{g}(x) = \dfrac{3x-2}{x^2-4} = \dfrac{3x-2}{(x-2)(x+2)}$ Domain $\{x \in \mathbb{R} : x \neq 2, x \neq -2\}$

$f \circ g(x) = f(g(x)) = 3(x^2-4) - 2$

$g \circ f(x) = g(f(x)) = g(3x-2) = (3x-2)^2 - 4$

- About Variables

ex
$$\begin{cases} f(g(x)) = 4x^2 - 8x \\ f(x) = x^2 - 4 \\ \text{What is } g(x)? \end{cases}$$

solution: Let $y = g(x)$ Get y in terms of x

$f(y) = 4x^2 - 8x \Rightarrow y^2 = 4x^2 - 8x + 4$

$f(y) = y^2 - 4$

$= (x-1)(4x-4)$

$y^2 = 4(x-1)^2$

$y = \pm 2|x-1|$

A solution: $\boxed{g(x) = 2(x-1)}$

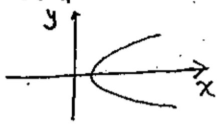
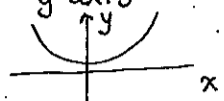
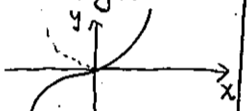
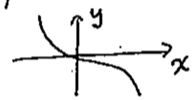
ex
$$\begin{cases} g(x) = 3x+2 \\ g(f(x)) = x \\ f(2) = ? \end{cases}$$

solution: $g(f(2)) = 2$

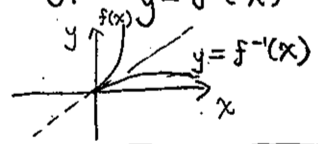
$g(f(2)) = 3f(2) + 2$

$\Rightarrow \boxed{f(2) = 0}$

V. Symmetry of Functions and Graphs

Symmetry	Name	Function Property	Graph Property
Symmetric About x-axis 	N/A	y is Not a function of x	(x,y) on graph $\Leftrightarrow (x,-y)$ on graph
Symmetric about y-axis 	<u>even</u>	$f(-x) = f(x)$ $y=f(x)$ is not invertible	(x,y) on graph $\Updownarrow$ $(-x,y)$ on graph
Symmetric about origin  	<u>odd</u>	$f(-x) = -f(x)$	(x,y) in graph $\Updownarrow$ $(-x,-y)$ in graph The graph in one quadrant is reflected across the y-axis & then reflected across the x-axis

- Symmetry about $y=x$ line. Means that if the graph contains the point (x,y), then the point (y,x) is also on the graph.
- Recall: The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ across the $y=x$ line.



ex Describe the symmetry of the function $f = \{(1,0), (-1,0), (3,0), (-3,0)\}$

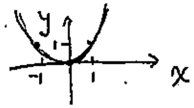
Solution: ① (x,y) on graph $\Rightarrow (-x,y)$ on graph $f(-x)=f(x)$

② $(-x,-y)$ $f(-x) = -f(x)$

f is ① symmetric about the y-axis (even function)
② symmetric about the origin (odd function)

- **ex** $f(x) = x^2$. Describe the symmetry.

Solution:



$f(-x) = (-x)^2 = x^2$

$f(-x) = f(x)$

$\boxed{f \text{ is even. (symmetric about y-axis)}}$

- **ex** Describe the symmetry of the graph of $x^4 + y^2 = 10$

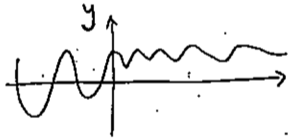
Solution: (x,y) "is on the graph" means $x^4 + y^2 = 10$

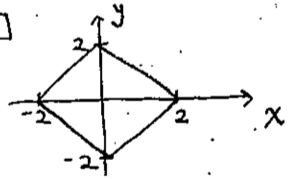
so, ① $(-x,y)$ is also on the graph since $(-x)^4 + y^2 = x^4 + y^2 = 10$

② $(-x,-y)$ " $(-x)^4 + (-y)^2 = x^4 + y^2 = 10$

③ $(x,-y)$ " $(x)^4 + (-y)^2 = x^4 + y^2 = 10$

The graph is ① symmetric about the y-axis (even)
② " the origin (odd)
③ " the x-axis

- ex  This graph has no symmetry

- ex  The graph of $|x|+|y|=2$ is symmetric w.r.t.
  - x-axis $(x,-y)$ OK
  - y-axis $(-x,y)$
  - origin $(-x,-y)$
  - $y=x$ line (y,x)

Note on functions

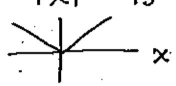
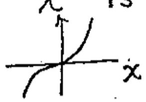
even + even = even	odd · even = odd
odd + odd = odd	even · even = even
odd + even = No conclusion	odd · odd = even

- Think of this as
  - pos + pos = pos, neg + neg = neg
  - neg · pos = neg, pos · pos = pos, neg · neg = pos.

- Example: $f(x)$ is odd, $g(x)$ is even $\Rightarrow$ $h(x)=fg(x)$ is odd

because
$$h(-x)=f(-x)g(-x)=-f(x)g(x)=-h(x)$$

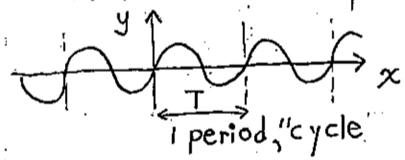
ex $f(x)=|x|x^{-3}$ is odd or even?

Solution: $|x|$ is even  x^3 is odd  $\Rightarrow \frac{1}{x^3}=x^{-3}$ is odd

$|x|$ is even, x^{-3} is odd $\Rightarrow$ $|x|x^{-3}$ is odd function
even × odd = odd

VI. Periodic Functions

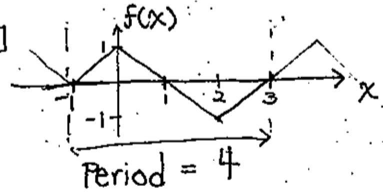
A function is periodic if it repeats itself in cycles.



$y=f(x)$

$\boxed{\text{Period } T = \text{smallest number such that } f(x+T)=f(x)}$

Note: $f(x+kT)=f(x)$, kT multiple of T

- ex 

Find $f(81)$

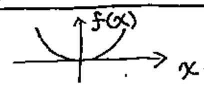
Solution: $f(81)=f(4\times 20+1)=f(1)=\boxed{0}$

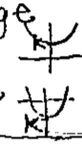
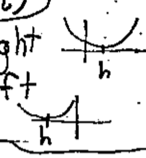
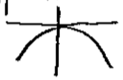
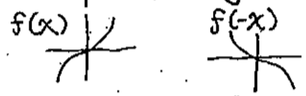
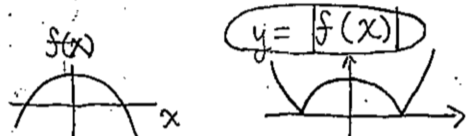
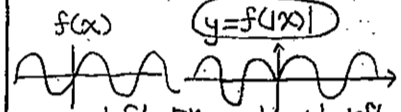
* Note: $\boxed{\text{If } f(x) \text{ has period } T \text{ then } g(x)=Af(Bx+C)+D \text{ has period } \frac{T}{B}}$

Why? Find the smallest number P s.t.
$$g(x+P)=g(x)$$
$$\begin{cases} g(x+P) = Af(Bx+C+BP)+D \\ g(x) = Af(Bx+C)+D \end{cases}$$
$g(x+P)=g(x)$ when $f(Bx+C+BP)=f(Bx+C)$

Since $f(z+T)=f(z)$, $BP=T \Rightarrow P=\frac{T}{B}$

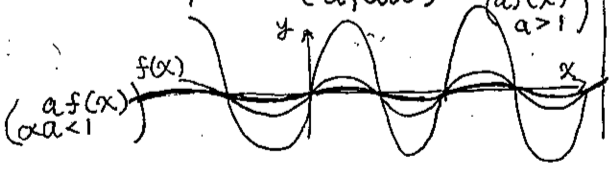
VII. Transformations

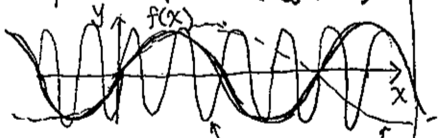


	Vertical	Horizontal
① Translation	$y=f(x)+k$ $\begin{cases} k=0 \text{ no change} \\ k>0 \text{ up} \\ k<0 \text{ down} \end{cases}$ 	$y=f(x-h)$ $\begin{cases} h>0 \text{ right} \\ h<0 \text{ left} \end{cases}$ 
② Reflection	$y=-f(x)$ flip across x-axis 	$y=f(-x)$ flip across y-axis 
	 $y=\lvert f(x)\rvert$ flip negative part up	 $y=f(\lvert x\rvert)$ ignore left. Flip right side left

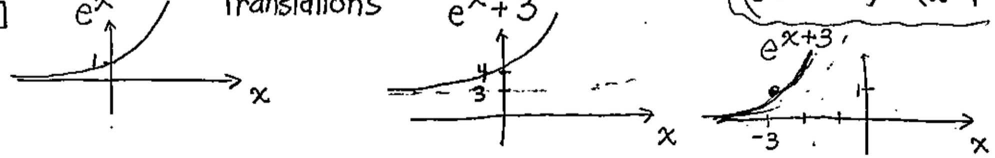
③

	Vertical	Horizontal
Stretch	$y = af(x)$ $\begin{cases} a>1 & \text{stretch} \\ 0<a<1 & \text{flatten}\end{cases}$	$y = f(ax)$ $a>1$ squeeze $0<a<1$ spread out
Graph of $y=f(x)$ has point (u,w)	Graph of $y = af(x)$ has point (u, aw)	Graph has point $(\frac{u}{a}, w)$



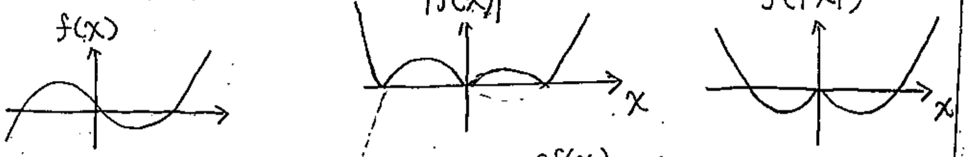


ex e^x Translations e^x+3 e^{x+3}

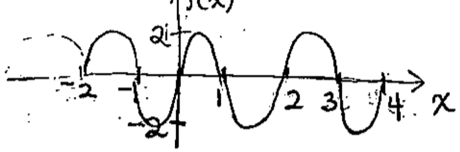


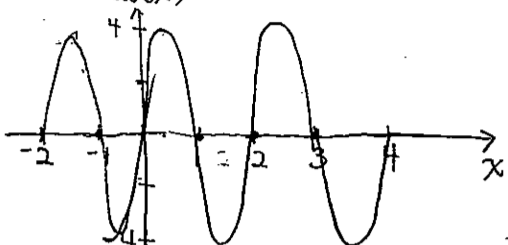
ex $\ln x$ Reflections $-\ln x$ $\ln(-x)$

ex Reflections $f(x)$ $|f(x)|$ $f(|x|)$

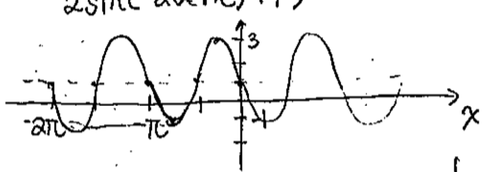


ex Stretches $f(x)$ $2f(x)$

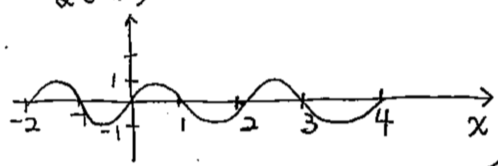




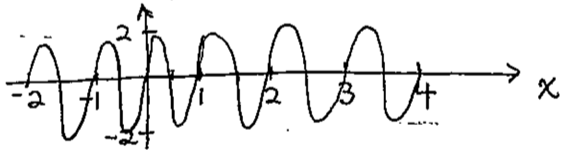
$2\sin(-2(x+\pi)+1)$

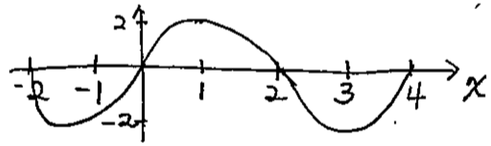


$\frac{1}{2}f(x)$



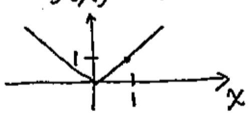
$f(2x)$ $f(x/2)$



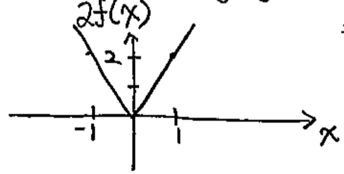


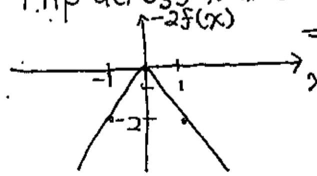
ex $f(x)=|x|$ Graph $y=-2|x|+3$

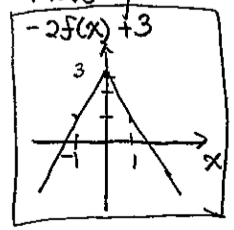
$y = -2f(x)+3$



stretch vertically by 2 $\Rightarrow$ Flip across x-axis $\Rightarrow$ Move up 3







★ Multiple horizontal translations:

(1) Factor out the constant in front of the x

(2) Go outside in. Order of transformations matter!

Given graph of $y=f(x)$, the graph of

$$\boxed{y = f(-ax+b) = f(-a(x-\tfrac{b}{a}))}$$

① Reflect across y-axis

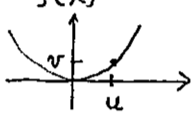
② Squeeze $a>1$ or stretch $a<1$

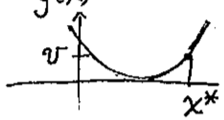
③ translate horizontally $\frac{b}{a}$ right (if $\frac{b}{a}>0$), $\frac{b}{a}$ left (if $\frac{b}{a}<0$)

Why? (u,v) is on the graph of $y=f(x)$, $(v=f(u))$

The graph of $y=g(x)=f(-ax+b)$ will have the point (x^*, v) if $g(x^*)=v$. Find x^*

$f(x)$ $g(x)$





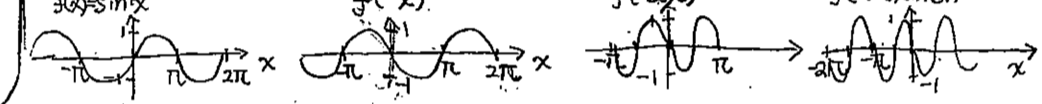
$f(-ax^*+b) = f(u)$

$\Rightarrow -ax^*+b = u$

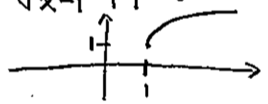
$x^* = -\frac{u}{a} - \frac{b}{a}$

ex Graph $2\sin(-2x-2\pi)+1 = 2f(-2(x+\pi))+1$

$f(x)=\sin x$ $f(-x)$ $f(-2x)$ $f(-2(x+\pi))$

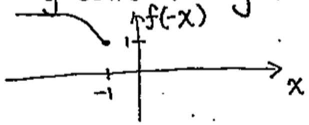


ex $\sqrt{x-1}+1=f(x)$



Sketch the graph of $y=\sqrt{-x-1}+1=f(-x)$

Solution: Just flip the graph of $y=f(x)$ across the y-axis to get the graph of $y=f(-x)$



Special Functions

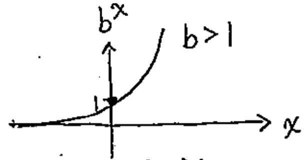
I. Exponential & Logarithmic Functions

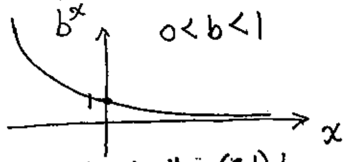
- Properties of exponents $b, r, s \in \mathbb{R}$

$b^r b^s = b^{r+s}$	$b^0 = 1$
$b^r / b^s = b^{r-s}$	$b^{-r} = \frac{1}{b^r}$
$(b^r)^s = b^{rs}$	$b^r q^r = (bq)^r$

Definition: Exponential function f with base b

base $b>0,\ b\neq 1$, $f(x)=b^x$





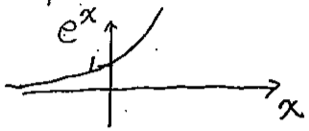
b^x is one-to-one

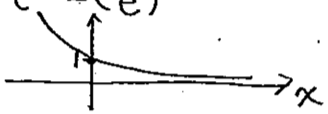
The inverse is $\log_b x$

$b>1$: thru $(0,1)$ increasing
asymptote: x-axis
$(\lim\limits_{x\to-\infty} b^x = 0)$
Exponential Growth

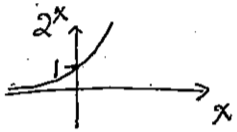
$0<b<1$: thru $(0,1)$ decreasing
Asymptote: x-axis
$(\lim\limits_{x\to\infty} b^x = 0)$
Exponential Decay

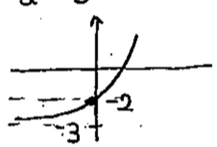
Natural Exponential function $e \approx 2.71828183\ldots$

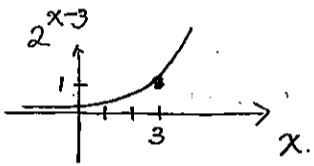




ex Sketch







- Properties of Logarithms

$b>0,\ b\neq 1$

$$\log_b(pq) = \log_b p + \log_b q$$
$$\log_b\left(\frac{p}{q}\right) = \log_b p - \log_b q$$
$$\log_b(p^s) = s\log_b p$$

1-1 Property

$$\log_b p = \log_b q \Leftrightarrow p = q$$

$$b^{\log_b p} = p \quad \text{for } p>0$$
$$\log_b(b^p) = p$$

★ Change of Base: Convenient $\log_{10}$, $\ln = \log_e$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$a, b, x > 0$
$a\neq 1,\ b\neq 1$

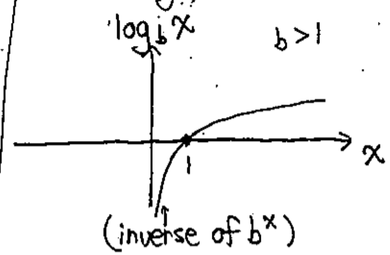
$$\frac{\log_b x}{\log_b y} = \frac{\log_a x}{\log_a y}$$

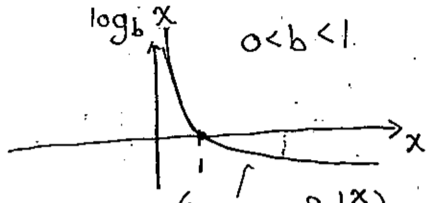
- Logarithm: Inverse of the exponential function
The exponent y such that b raised to this power gives x

$y=\log_b x \Longleftrightarrow b^y = x$ ★Domain $x>0$ ★

$b>0,\ b\neq 1$

- Notice: Graph of inverse is reflection about $y=x$ axis





$b>1 \Rightarrow \log_b x$ increasing
$(1,0)$ on graph
Asymptote: y-axis
$\left(\lim\limits_{x\to 0} \log_b x = -\infty\right)$
$(b^{-\infty}=0)$

$0<b<1 \Rightarrow \log_b x$ decreasing
$(1,0)$ on graph
Asymptote: y-axis
$\left(\lim\limits_{x\to 0} \log_b x = +\infty\right)$
$b^{\infty} = 0$

ex $25^{1/2}=x \Leftrightarrow \frac{1}{2}=\log_{25}x$

ex $\log_b \frac{x^2\sqrt{y}}{z^5} = 2\log_b x + \frac{1}{2}\log_b y - 5\log_b z$

ex $4\log_b(x+2) - 3\log_b(x-5) = \log_b \frac{(x+2)^4}{(x-5)^3}$

ex $\frac{\ln 2}{\ln 8} = \frac{\log_2 2}{\log_2 8} = \frac{\log_2 2^1}{\log_2 2^3} = \boxed{\frac{1}{3}}$

ex $\log_2 m=\sqrt{7}$, $\log_7 n=\sqrt{2}$, $mn=?$

Soln: $m=2^{\sqrt{7}}$

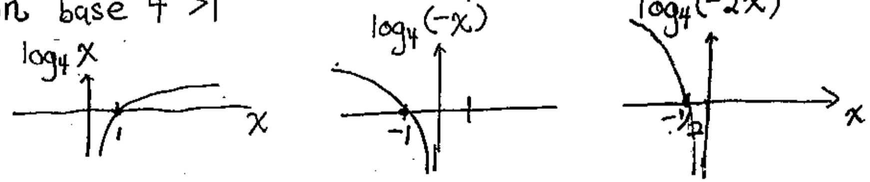
$n=7^{\sqrt{2}}$

$mn=2^{\sqrt{7}}\,7^{\sqrt{2}} \approx \boxed{98}$

ex Evaluate $\log_3 \frac{18}{3} = \frac{\log_{10}(18/3)}{\log_{10}3} = 1.631$

ex Sketch $y=\log_4(-2x)$

solution base 4 >1



ex Solve $5^x=40$ Soln: $\ln 5^x = \ln 40$

$x\ln 5 = \ln 40 \Rightarrow x = \frac{\ln 40}{\ln 5} = \boxed{2.29}$

ex Solve $\log(2x) - \log(x-2.4) = 1$ $x>0,\ x>2.4$

Soln: $\log\left(\frac{2x}{x-2.4}\right)=1$

$\frac{2x}{x-2.4} = 10^1$ $x \neq 3$

$2x = 10x-24$

$24 = 8x$

$x=3$

$\boxed{\text{No Solution!}}$

ex Solve $\frac{2^x+2^{-x}}{2}=3$ (b^x+b^{-x})

$2^x+2^{-x}=6$

Let $u=2^x$

$u+\frac{1}{u}=6$

$u^2+1=6u$

$u^2-6u+1=0$

$u=\frac{6\pm\sqrt{36-4}}{2} = 3\pm 2\sqrt{2}$

$\Rightarrow 2^x = 3\pm2\sqrt{2}$

$x=\log_2(3\pm2\sqrt{2})$

$=\frac{\ln(3\pm2\sqrt{2})}{\ln 2}$

$=\boxed{\pm 2.543}$

ex Compound interest = interest paid on principal (amount you have before) + interest paid on interest earned previously

Balance = total amount

$r=0.12$

You invest $P=\$1000$ at 12% annual interest compounded annually. What's the balance after 3 years?

After 1 year: $P+rP=(1+r)P$

2 years: $(1+r)P+r(1+r)P$

$=(1+r)^2P$

N years $(1+r)^N P \Rightarrow (1.12)^3\,\$1000 = \boxed{\$1404.928}$

ex $\log_{10}23=z$, $\log 2300=?$ Soln $\log(23\times100)=\log 23+\log 10^2 = \boxed{z+2}$

ex $\log_b 2 = x$

$\log_b 3 = y$

$\log_b 18 = ?$

Soln: $\log_b(2\times3\times3) = x+y+y = \boxed{x+2y}$

ex Solve $\log_b(x-5) = \log_b x + \log_b 5$

Soln: $x-5>0$, $x>0$, $5>0$ $\boxed{\text{No Solution!}}$

$x>5$ Restriction

$\log_b\left(\frac{x-5}{x}\right) = \log_b 5 \Rightarrow x-5=5x$

$-5=4x$

$x=-5/4$

ex $\log_{27}\sqrt{54} - \log_{27}\sqrt{6} = ?$

$=\log_{27}\sqrt{\frac{54}{6}} = \log_{27}3 = \frac{\log_3 3}{\log_3 27} = \frac{\log_3 3^{1}}{\log_3 3^{3}} = \boxed{\frac{1}{3}}$

ex Solve: $\log_8 3 = x\log_2 3$ Soln: $x=\frac{\log_8 3}{\log_2 3} = \frac{\log_2 3/\log_2 8}{\log_2 3}$

$=\frac{1}{\log_2 2^3} = \boxed{\frac{1}{3}}$

ex $\log_{10} m = \frac{1}{2}$

$\log_{10} 10m^2 = ?$ Soln: $\log_{10}10 + 2\log_{10}m = 1+2\left(\frac{1}{2}\right) = \boxed{2}$

ex $a=\log_b 5$

$c=\log_b 2.5$

$5^x = 2.5$

$x=?$

in terms of a & c

Soln: $x\log_b 5 = \log_b 2.5$

$xa=c$

$\boxed{x=c/a}$

ex $f(x)=\log_2 x$

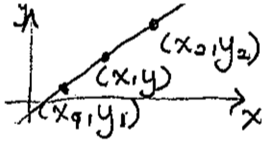
$f\left(\frac{2}{x}\right)+f(x)=?$

Soln: $\log_2\left(\frac{2}{x}\right)+\log_2 x$

$=\log_2 2 - \cancel{\log_2 x} + \cancel{\log_2 x} = \boxed{1}$

(4) Polynomials – Linear Functions / Equations

- straight line
- Given $\boxed{\text{Point \& Slope or 2 points}}$



- Slope $= \frac{\text{rise}}{\text{run}}$ is constant

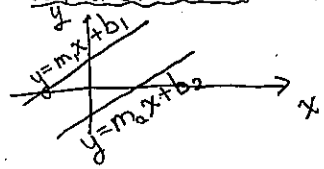
Given (x_1,y_1)
(x_2,y_2) or m
Variables (x,y) on the line must satisfy

$\boxed{m = \frac{y-y_1}{x-x_1} \quad \frac{y_2-y_1}{x_2-x_1}}$ slope is constant

- $\boxed{\begin{array}{l}\text{Point-Slope Form: } y-y_1 = m(x-x_1) \\ \text{Standard Form: } Ax+By=C, \ (A,B,C \text{ constants}) \\ \text{Slope-Intercept Form: } y = mx+b\end{array}}$

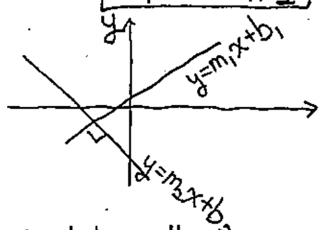
slope y-intercept

- Parallel Lines: same slope

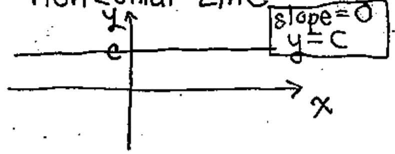


$m_1=m_2$
$b_1 \neq b_2$

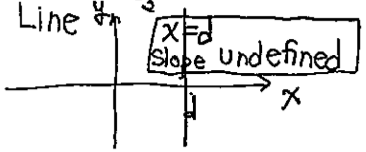
- Perpendicular Lines $\boxed{m_1 = -\frac{1}{m_2}}$



- Horizontal Line $\boxed{\begin{array}{l}\text{slope}=0\\ y=c\end{array}}$



Vertical Line $\boxed{\begin{array}{l}x=d\\ \text{Slope undefined}\end{array}}$



[ex] Write the equation of a line through $(1,-3),(-4,-2)$

Soln: $m = \frac{-2-(-3)}{-4-1} = \frac{1}{-5} = \frac{y+3}{x-1}$, or $-\frac{1}{5} = \frac{y+2}{x+4}$

(1) Point-Slope form (not unique)
$y+3 = -\frac{1}{5}(x-1)$, $y+2 = -\frac{1}{5}(x+4)$
same line

(2) Standard form (not unique)
$x+5y = -14$ ← same line
$2x+10y = -28$

(3) Slope-intercept form is unique: $y = \underbrace{-\frac{1}{5}}_{\text{slope}}x \underbrace{-\frac{14}{5}}_{\text{y-intercept}}$

[ex] f is a linear function s.t. $f(7)=5$, $f(12)=-6$ and $f(x)=23.7$, What is the value of x?

Soln: $m = \frac{-6-5}{12-7} = -\frac{11}{5} = \frac{23.7-5}{x-7} \Rightarrow \boxed{x=-1.52}$

- [ex] Find the line through $(1,7)$ and parallel to $\underset{m_1}{3x+5y=8}$

Soln: Slope $3x+5y=8$
$y = -\frac{3}{5}x+\frac{8}{5} \Rightarrow \boxed{-\frac{3}{5} = \frac{y-7}{x-1}}$
$m_1 = -\frac{3}{5}$

- [ex] ① $\pi x+\sqrt{2}y+\sqrt{3}=0$ is perpendicular to ② $ax+3y+2=0$ Find a

Soln: ① Slope? $y = -\frac{\pi}{\sqrt{2}}x - \sqrt{3/2}$
$m_1 = -\frac{\pi}{\sqrt{2}}$

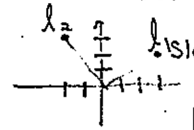
② $y = -\frac{a}{3}x - \frac{2}{a}$

$\Rightarrow -\frac{a}{3} = +\frac{\sqrt{2}}{\pi}$

$\boxed{a = -\frac{3\sqrt{2}}{\pi}}$

- [ex] Point $P(3,2)$ is rotated $90°$ counterclockwise. What are the new coordinates?

Soln:



$l_2 \perp l_1 \Rightarrow m_2 = -\frac{3}{2} = \frac{\text{up } 3}{\text{left } 2}$

$\boxed{(-2,3)}$ The distance from the origin is still $\sqrt{2^2+3^2}$

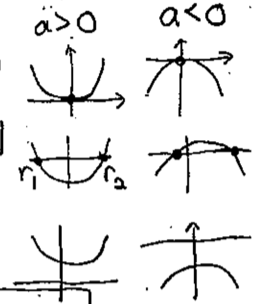
(III) Quadratic Functions (Parabolas)

★ $\boxed{\text{Vertex } \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)}$

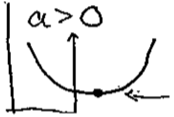
$\boxed{\begin{array}{l} y = f(x) = ax^2+bx+c \\ \quad = a(x-r_1)(x-r_2) \end{array}}$

$r_1, r_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

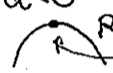
Discriminant $D = b^2-4ac$
$=0 \Rightarrow r_1=r_2$ real
$>0 \Rightarrow r_1 \neq r_2$, real
$<0 \Rightarrow r_1$ & r_2 are complex conjugates



$\boxed{\begin{array}{l} r_1+r_2 = -\frac{b}{a} \\ r_1 \cdot r_2 = \left(-\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{(2a)^2} \end{array}}$



Range $\left[f\left(-\frac{b}{2a}\right), \infty\right)$
minimum



Range: $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$
maximum

$\boxed{ex}$ Solve $3x^2+10x=8$. Factoring Method.

Soln: $3x^2+10x-8=0$ $\begin{matrix}1 & 4\\ 3 & -2\end{matrix}$ $(x+4)(3x-2)=0$ $\boxed{x=-4 \text{ or } x=\frac{2}{3}}$

$\boxed{ex}$ Method of Taking Square Roots

Solve $(x-3)^2=-28$

Soln: $(x-3)=\pm\sqrt{-28}=\pm\sqrt{2^2 i^2 7}=\pm 2i\sqrt{7}$

$\Rightarrow \boxed{x = 3+2i\sqrt{7} \text{ or } 3-2i\sqrt{7}}$ complex conjugate roots

Notice $(x-3)^2+28=(x-3-2i\sqrt{7})(x-3+2i\sqrt{7})$

$\boxed{ex}$ Complete the Square (or use quadratic formula)

$2x^2+8x-15=0$

$2(x^2+4x-\frac{15}{2})=0$

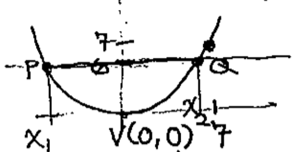
$x^2+2(2)x+\boxed{2^2}=\frac{15}{2}+\boxed{2^2}$

$(x+2)^2=\frac{23}{2}$

$x+2=\pm\sqrt{\frac{23}{2}} \Rightarrow \boxed{x=-2\pm\sqrt{\frac{23}{2}}}$

• $\boxed{ex}$ Parabola with vertex at origin passes through $P(7,7)$, & intersects line $y=6$ at 2 points. What is the length of the segment joining the 2 points?

Soln: Must open up $y=\underbrace{ax^2+bx+c}_{f(x)}=6$ at 2 points



$V(-\frac{b}{2a}, f(-\frac{b}{2a}))$

$-\frac{b}{2a}=0 \Rightarrow \boxed{b=0}$

$0=f(0)=c \Rightarrow \boxed{c=0}$

$y=ax^2$

$7=a7^2 \Rightarrow \boxed{a=\frac{1}{7}}$

$x_2=?$ $ax_2^2=6$

$\frac{1}{7}x_2^2=6 \Rightarrow x_2=\sqrt{42}$ $2|x|=\boxed{2\sqrt{42}}$

$x_1=-\sqrt{42}$ length of $\overline{PQ}$

• $\boxed{ex}$ Sum of the roots of $(x-\sqrt{2})(x^2-\sqrt{3}x+\pi)=0$?

roots $r_1, r_2=\frac{\sqrt{3}\pm\sqrt{3-4\pi}}{2}$

$\Rightarrow r_1+r_2=\frac{\sqrt{3}}{2}\times 2=\sqrt{3}$

$\Rightarrow \sqrt{2}+r_1+r_2=\boxed{\sqrt{2}+\sqrt{3}}$

$\boxed{ex}$ For what positive values of n are the zeros of $P(x)=5x^2+nx+12$ in ratio $2:3$?

Soln: $5\underbrace{(x^2+\frac{n}{5}x+\frac{12}{5})}_{(x-r_1)(x-r_2)}=0$ $\begin{matrix}ax^2+bx+c\\ 1 \quad \frac{n}{5} \quad 12/5\end{matrix}$

$(x-r_1)(x-r_2)=0$ $r_1, r_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$r_1:r_2=2:3$

$\Rightarrow r_1=2k$

$r_2=3k$

k=constant integer

$\Rightarrow$ ① $r_1+r_2=-\frac{b}{a}$

$5k=-\frac{n}{5}$

② $r_1r_2=(\frac{b}{2a})^2+\frac{b^2-4ac}{(2a)^2}=\frac{c}{a}$

$6k^2=\frac{12}{5} \Rightarrow k=\pm\sqrt{\frac{2}{5}}$

①&② $\Rightarrow n=25k=\pm 25\sqrt{\frac{2}{5}} \approx \boxed{15.8}$

• $\boxed{ex}$ Solve $x^2-1<0$

soln: $x^2<1$

$\sqrt{x^2}<\sqrt{1}$

$|x|<1$

$\boxed{-1<x<1}$

• $\boxed{ex}$ Solve $-(x-3)(x+5)\geq 0$

$\boxed{-5\leq x\leq 3}$

• $\boxed{ex}$ What is the Range of $f(x)=|-x^2-8|$?

Soln: $f(x)=|x^2+8|=x^2+8$ $\boxed{[8,\infty)}$

★Short cuts

• $\boxed{ex}$ $ab=3$, $bc=5/9$, $ac=15$ $abc=?$

Soln: $ab\,bc\,ac=3(\frac{5}{9})15$

$(abc)^2=25$

$\boxed{abc=\pm 5}$

• $\boxed{ex}$ $3s+5t=10$

$2s-t=7$

Find $\frac{1}{2}s+3t$? Soln: $3s+5t=10$

$-(2s-t=7)$

$s+6t=3 \Rightarrow \frac{1}{2}s+3t=\boxed{\frac{3}{2}}$

• $\boxed{ex}$ $n-m=-3$

$n^2-m^2=24$

$n+m=?$

Soln: $\underbrace{(n-m)}_{-3}(n+m)=24$

$\Rightarrow n+m=\frac{24}{-3}=\boxed{-8}$

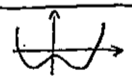
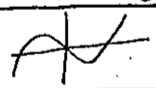
Higher-Order Polynomials

- Polynomial Behavior

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

- Leading term dominates behavior for $|x|$ big ($a_n \neq 0$)
- The degree of the polynomial is n

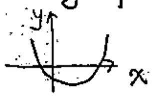
End Behavior

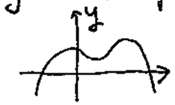
	n even	n odd
$a_n > 0$		
$a_n < 0$		

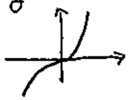
- Some facts on polynomials:

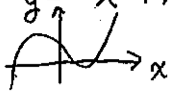
1. Polynomials are continuous (The graph can be drawn without removing pencil from paper.)
2. If the largest exponent is an even number, both ends of the polynomial either both go up or both go down.
3. If the largest exponent is an odd number, the ends of the graph go in opposite directions.

ex


$y = x^2 + \ldots$


$y = -x^4 + \ldots$

$y = x^3 + \ldots$


$y = -x^5 + \ldots$


4. If all the exponents are even numbers, the polynomial is an even function

ex $P(x) = 3x^4 + 2x^2 - 8 \Rightarrow P(-x) = P(x)$

5. If all the exponents are odd numbers and there is no constant term, then the polynomial is an odd function.

ex $f(x) = 5x^5 + 2x^3 - 3x$

$f(-x) = 5(-x)^5 + 2(-x)^3 - 3(-x)$

$= -5x^5 - 2x^3 + 3x$

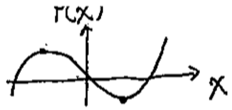
$= -f(x)$

so f is odd function

But $g(x) = 3x^3 + 1$ is NOT an odd function.

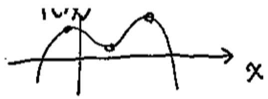
★ 6. $\boxed{\text{A polynomial of degree } n \text{ can have at most } (n-1) \text{ maxima or minima}}$

ex



2 extrema

$\Rightarrow P(x)$ has degree ≥ 3



3 extrema $\Rightarrow P(x)$ has degree ≥ 4

7. c is called a "zero" or "root" of polynomial $P(x)$ if $P(c) = 0$

★ A polynomial $P(x)$ with real coefficients and degree n must have n zeros (roots). The roots are real or in complex conjugate pairs. If a zero occurs m times, it has multiplicity m

ex $f(x) = (x-5)(x+3)^4(x^2+6)$

Degree: 7

Real zero	Multiplicity
5	1
−3	4

Complex zeros: $x^2 + 6 = 0$

$x = i\sqrt{6}$ or $-i\sqrt{6}$

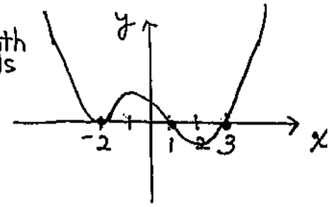
conjugate pair

★ If a real zero c has even multiplicity, then $P(x)$ is tangent to the x-axis at $x = c$.
If a real zero c has odd multiplicity, then $P(x)$ crosses the x-axis at $x = c$.

- ex Sketch $P(x) = (x+2)^2(x-1)(x-3)$

degree: 4. Leading coefficient: $+1 \Rightarrow$ both ends up

real zero	multiplicity		
−2	2	→ even →	tangent
1	1	odd	x-crossing
3	1	odd	x-crossing



Polynomial Division

ex

$$
\begin{array}{r}
x^2-3x+17 \leftarrow \text{quotient} \\
\underbrace{x^2+3x-5}_{\text{divisor}} \overline{)\, x^4+0x^3+3x^2-6x-10} \leftarrow \text{Dividend} \\
\underline{x^4+3x^3-5x^2} \\
-3x^3+8x^2-6x-10 \\
\underline{-3x^3-9x^2+15x} \\
17x^2-21x-10 \\
\underline{17x^2+51x-85} \\
-72x+75 \leftarrow \text{remainder}
\end{array}
$$

$$\frac{x^4+3x^2-6x-10}{x^2+3x-5} = (x^2-3x+17) + \frac{-72x+75}{x^2+3x-5}$$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$P(x) = D(x)\cdot Q(x) + R(x)$$

- ex Perform the indicated division

$$\frac{x^4-4x^2+7x+15}{x+4} = ? \quad \leftarrow (x^3-4x^2+12x-41) + \frac{179}{x+4}$$

Soln:

$$
\begin{array}{r}
x^3-4x^2+12x-41 \\
x+4 \overline{)\, x^4-4x^2+7x+15} \\
\underline{x^4+4x^3} \\
-4x^3-4x^2+7x+15 \\
\underline{-4x^3-16x^2} \\
12x^2+7x+15 \\
\underline{12x^2+48x} \\
-41x+15 \\
\underline{-41x-164} \\
179
\end{array}
$$

Notice that division by $x+4$ gives a constant (179) remainder.

$$\frac{P(x)}{x-r} = Q(x) + \frac{c}{x-r}$$

$$P(x) = (x-r)Q(x) + c$$

$$P(r) = 0 + c$$

$c=P(r)$. The remainder of division by $x-r$ is $P(r)$.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$
$$= a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$
$$r_1, r_2, \dots r_n \in \mathbb{C}$$

Theorems on Roots & Factoring

① The Remainder Theorem. If a polynomial $P(x)$ is divided by $x-r$, then the remainder is $P(r)$.

ex Polynomial $P(x) = 2x^3+3x^2+2x-2$ is divided by $x-\frac{1}{2}$ with a remainder of 0. $P(\frac{1}{2}) = ?$

Soln: $P(\frac{1}{2}) = \text{remainder} = 0$

★ ② The Factor Theorem zeros / Roots $\Leftrightarrow$ factors

$$\boxed{P(x) = \underbrace{(x-r)}_{\text{factor}}Q(x) \Leftrightarrow P(r)=0}$$

ex Is there a common factor in the numerator & denominator?

$$\frac{x^3-8}{x^4-16} = \frac{P(x)}{Q(x)}.$$

$P(2)=0 \Leftrightarrow (x-2)$ is a factor of $P(x)$

$Q(2)=0 \Leftrightarrow (x-2)$ 〃 $Q(x)$

So there is a common factor of $x-2$

ex Factor ax^2+bx+c

Soln: Roots $r_1, r_2 = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\Rightarrow ax^2+bx+c = a(x-r_1)(x-r_2)$$

ex What needs to be satisfied for $P(x) = ax^2+bx+c$ to be divisible by $x+k$?

solution: $P(-k)=0$

- ex Is $x-99$ a factor of $P(x) = x^4-100x^3+97x^2+200x-198$?

Soln: $P(99) = 0$. yes.

★ The Fundamental Theorem of Algebra (by Gauss). A polynomial $P(x)$ with degree $n\geq 1$ and complex coefficients has at least one complex zero.

(Note a complex number is $a+bi \in \mathbb{C}$, where a and are real. If $b=0$, then $a+0i$ is just a real number. $\mathbb{R} \subset \mathbb{C}$)

- $P(x) = (x-c)\underline{Q(x)} = (x-c)(x-d)\underbrace{G(x)}_{\text{has a zero...}}$

  - 1 degree less
  - has at least one zero

⟸ Corollary: A Polynomial of degree n with complex coefficients has exactly n complex zeros (roots)

★③ Conjugate Pair Theorem. $P(z)$ has real coefficients.
If $a+bi$ ($b\neq0$) is a zero (root) of $P(z)$, then $a-bi$ is also a zero.

★④ Descartes' Rule of Signs
- $P(x)$ is a polynomial with real coefficients
- Arrange the terms in decreasing powers of x

The number of positive real zeros = The number of alternating signs in the coefficients of $P(x)$ OR this number decreased by an even integer.

The number of negative real zeros = The number of alternating signs in the coefficients of $P(-x)$ OR this number decreased by an even integer.

- ex Describe the possible roots of $P(x)=18x^4+7x^2-5-2x^3+8x$ in terms of the number of positive or negative real zeros and the number of complex zeros.

Soln: • $P(x)$ must have 4 zeros
• Descarte's Rule of Signs
(1) $P(x)=18x^4-2x^3+7x^2+8x-5$
3 alternating signs ⇒ 3 real positive roots or 1 real positive rule
(2) $P(-x)=18x^4+2x^3+7x^2-8x-5$
1 alternating sign ⇒ 1 real negative root

Possibilities (4 zeros total)

# positive real zeros	# negative real zeros	# $\mathbb{C}$ conjugate pairs
3	1	0
1	1	1

ex Possibilities for the roots of $P(x)=4x^2-x^5+1$
Soln: $P(x)=-x^5+4x^2+1$ ⇒ 1 positive real zero
$P(-x)=x^5+4x^2+1$ ⇒ 0 negative real zeros
2 complex conjugate pairs
Total: 5 zeros

ex $P(x)=3x^5-36x^4+2x^3-8x^2+9x-338$.
Given that $3+2i,\ 2-3i,\ 5$ are zeros, what are the other roots?
Solution: There are 5 roots total. Complex roots must come in conjugate pairs. The other two roots must be $\boxed{3-2i \text{ and } 2+3i}$

ex Find all the zeros of $P(x)=x^4-4x^3+14x^2-36x+45$ given that $2+i$ is a zero.
Solution: There are 4 roots. Two of them are $2+i,\ 2-i$.
Find the other two, which must be roots of $Q(x)$.

$$P(x)=\underbrace{(x-(2+i))(x-(2-i))}_{x^2-4x+(2^2+1^2)}Q(x)$$

$$Q(x)=\frac{x^4-4x^3+14x^2-36x+45}{x^2-4x+5}=x^2+9$$

By long division:

$$\begin{array}{r} x^2\qquad\qquad\;\; +9 \\ x^2-4x+5\,\big)\,\overline{x^4-4x^3+14x^2-36x+45} \\ x^4-4x^3+5x^2\qquad\qquad\quad \\ \hline 9x^2-36x+45 \\ 9x^2-36x+45 \\ \hline 0 \end{array}$$

$x^2+9=0$
$\Rightarrow x^2=-3^2$
$x=\pm\sqrt{3^2 i^2}$
$=\pm 3i$
$Q(x)=x^2+9$
$=(x-3i)(x+3i)$

$\Rightarrow P(x)=(x-(2+i))(x-(2-i))(x-3i)(x+3i)$
$\boxed{\text{The roots are } 2+i,\ 2-i,\ 3i,\ -3i}$ Two complex conjugate pairs.

- Note: We say a quadratic polynomial is irreducible if it has no real zeros.
ax^2+bx+c
$D=b^2-4ac<0$
discriminant

$x^2+9=(x-3i)(x+3i)$
irreducible, not real

- ex $x-1$ and $x-2$ are factors of $x^3-3x^2+2x-4b$. $b=?$
Solution: Let $P(x)=x^3-3x^2+2x-4b$
$P(1)=0=1-3+2-4b \Rightarrow b=0$
$P(2)=0$ is extra information

- ex Write the polynomial with lowest degree if two of its roots are -1 and $1+i$.
Solution: $(x-(-1))(x-(1+i))(x-(1-i))=(x+1)(x^2-2x+(1^2+1^2))$
$=(x+1)(x^2-2x+2)=\boxed{x^3-x^2+2}$

(5) Rational Zero Theorem

$P(x) = a_n x^n + \dots + a_1 x + a_0$ with integer coefficients.

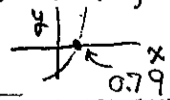
If p/q (no common factors) is a rational zero of $P(x)$, then p is a factor of a_0 (the constant) and q is a factor of a_n (the leading coefficient).

ex $P(x) = 3x^3 + 2x^2 + 4x - 6$. Find all possible rational zeros.

$q: \pm 1, \pm 3$ $p: \pm 1, \pm 6, \pm 2, \pm 3$

$\frac{p}{q}: \pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 6, \left(\pm\frac{6}{3}\right), \pm 3, \left(\pm\frac{3}{3}\right)$

★ Note: This is a list of possible rational zeros. It may be that $P(x)$ has no rational zeros at all; as in this example, the only real zero is irrational!

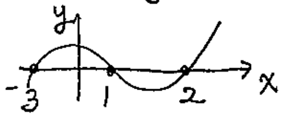


by graphing calculator.

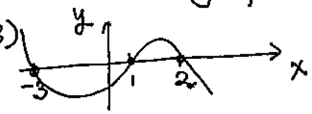
More Polynomial Practice

ex Find a polynomial that matches the graph or description.

A)



B)



Solution: $y = (x+3)(x-1)(x-2)$ Soln: $y = -5(x+3)(x-1)(x-2)$

C) $P(x)$ has degree 4, zeros $2i$, $3-7i$

Solution: $P(x) = a(x-2i)(x+2i)(x-(3-7i))(x-(3+7i))$

$= a(x^2 + 2^2)(x^2 - 6x + (3^2 + 7^2))$

$= a(x^2+4)(x^2 - 6x + 58)$

$= \boxed{a(x^4 - 6x^3 + 62x^2 - 24x + 232)}$

a is any real number. The graph of $y = P(x)$ does not cross the x-axis.

ex $P(x) = ax^4 + x^3 - bx^2 - 4x + c$.

If $\lim_{x\to\infty} P(x) = \infty$, then $\lim_{x\to-\infty} P(x) = ?$

solution: degree 4. If the right end goes up, $a > 0$, and the left end also goes up. $\lim_{x\to-\infty} P(x) = \boxed{\infty}$

ex Which is an odd function?

I. $f(x) = 3x^3 + 5$

II. $g(x) = 4x^6 + 2x^4 - 3x^2$

III. $h(x) = 7x^5 - 8x^3 + 12x$

Solution:

$f(-x) \neq f(x)$ or $-f(x)$

$g(-x) = g(x)$ even

$h(-x) = -h(x)$ odd

Only III. is odd function

ex Can an even function be odd too?

solution: $f(x)$ is even means $f(-x) = f(x)$ for all x

f is odd means $f(-x) = -f(x)$

f is both odd & even means $f(-x) = f(x)$, $f(-x) = -f(x)$

$\Rightarrow f(x) = -f(x)$ for all x.

$\boxed{f(x) = 0}$

ex Find x such that $x^2 - 3x - 4 < 0$.

Solution: $(x-4)(x+1) < 0$

$\boxed{-1 < x < 4}$

ex Solve $x^5 - 3x^3 + 2x^2 - 3 > 0$

Solution: use your graphing calculator

$\boxed{(-1.9, -0.8) \cup (1.5, \infty)}$

ex How many integers are in the solution set of $x^2 + 48 < 16x$?

solution: $x^2 - 16x + 48 < 0$

$(x-4)(x-12) < 0$

$4 < x < 12$. 5, 6, 7, 8, 9, 10, 11 are integers in the solution set. There are $\boxed{7}$ integers that satisfy the inequality.

(VI) Rational Functions, Limits, Asymptotes

$$F(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$$

F is a rational function, the quotient of polynomials

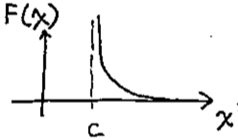
Domain of $F(x)$ is $\{x \in \mathbb{R} : Q(x) \neq 0\}$

ex $\dfrac{x^2 - x - 5}{x(2x-5)(x+3)}$ Domain $= \{x \in \mathbb{R} : x \neq 0, 5/2, -3\}$

II. Rational Function Asymptotes

1. Vertical Asymptote $(x=c)$

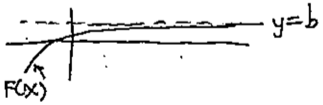
$$\lim_{x\to c^+} F(x) = \infty \text{ or } -\infty$$
$$\lim_{x\to c^-} F(x) = \infty \text{ or } -\infty$$

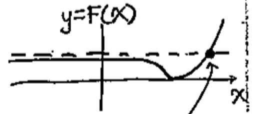


$x=c$ is a vertical asymptote of the curve $y=F(x)$ if $F(x)$ shoots up to positive or negative infinity as x approaches c from the left or right.

* F cannot cross a vertical asymptote since $F(c)$ is undefined (usually division by zero)

2. Horizontal Asymptote $(y=b)$

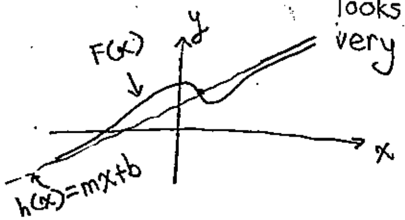


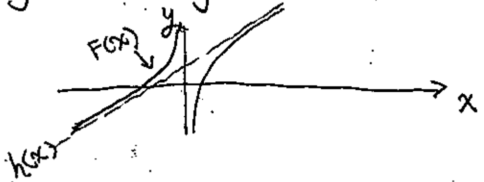


F can cross a horizontal asymptote

$$\lim_{x\to\infty} F(x)=b$$
or
$$\lim_{x\to-\infty} F(x)=b$$

3. Slant Asymptote: $(h(x) = mx+b)$ The line $h(x)=mx+b$ is a slant asymptote of $F(x)$ if the graph of $y=F(x)$ looks like the line $y=h(x)$ when x is very large or very small.





As $x\to+\infty$, $F(x)\to h(x)$

or As $x\to-\infty$, $F(x)\to h(x)$

① Finding Vertical Asymptotes (where denominator $=0$)

$F(x) = \dfrac{P(x)}{Q(x)}$ Simplify to no common factors.

$c\in\mathbb{R}$ is a zero of $Q(x)$ $(Q(c)=0)$ means that $x=c$ is a vertical asymptote of the rational function $F(x)$.

② Theorem on Horizontal Asymptotes (Behavior of $F(x)$ when $|x|$ is big) Look at the leading terms

$$F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}$$

	Horizontal Asymptote
$n<m$	x-axis $(y=0)$
$n=m$	$y=a_n/b_m$
$n>m$	no horizontal asymptote

③ Theorem on Slant Asymptotes

$$F(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_1 x + b_0} \qquad n=m+1$$

When the numerator is one degree higher than the denominator, $F(x)$ has a slant asymptote. Use long division to get the line. The remainder $r(x)$ will have degree one less than $Q(x)$, so when $|x|$ is large $r(x)/Q(x)$ is insignificant.

$$F(x) = \frac{P(x)}{Q(x)} = (mx+b) + \frac{r(x)}{Q(x)}$$ ← degree of $r(x)$ is one less than $Q(x)$

- As $x\to\infty$, $-\infty$, $F(x)\to \underset{\text{line}}{mx+b}$ since $\dfrac{r(x)}{Q(x)}\to 0$
- The slant asymptote is $y=mx+b$.

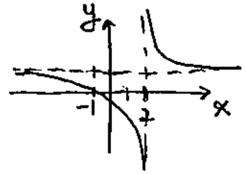
Examples of Rational Functions

ex $G(x) = \dfrac{x+1}{x-2}$

- Vertical Asymptote: $x=2$
- Horizontal ": $y=1$

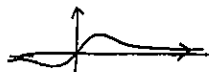
$$\left(\lim_{x\to\infty}\frac{x+1}{x-2} = \lim_{x\to\infty}\frac{1+\frac{1}{x}}{1-\frac{2}{x}} = 1\right)$$

- Slant Asymptote: none (numerator degree is not one greater than denominator's)



ex Find the horizontal asymptotes

a) $f(x) = \dfrac{2x+3}{x^2+1}$ Solution: $\lim_{x\to\infty}\dfrac{2x}{x^2} = \lim_{x\to\infty}\dfrac{2}{x} = 0$, $\lim_{x\to-\infty}\dfrac{2x}{x^2} = 0$ $\boxed{y=0}$



b) $g(x) = \dfrac{4x^2+1}{3x^2}$ Solution: $\boxed{y=4/3}$

c) $h(x) = \dfrac{x^3+1}{x-2}$ Solution: $\boxed{\text{No horizontal asymptote}}$ degree of numerator > deg. of denominator

ex Find the asymptotes of $f(x) = \dfrac{2x^3+5x^2+1}{x^2+x+3}$.

Solution: Vertical? $x^2+x+3=0$ for a real number x?
discriminant $D = 1^2-4(1)(3) < 0$. No real zeros.
No vertical asymptote.

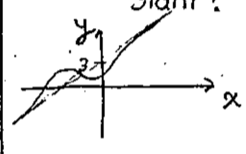
Horizontal? None.

Slant? Yes.

$$\begin{array}{r} 2x+3 \\ x^2+x+3\,\overline{)\,2x^3+5x^2+1} \\ \underline{2x^3+2x^2+6x} \\ 3x^2-6x+1 \\ \underline{3x^2+3x+9} \\ -9x-8 \end{array}$$

$$f(x) = (2x+3) - \frac{9x+8}{x^2+x+3}$$
insignificant when $|x|\to\infty$

$\boxed{y=2x+3}$ Slant asymptote



Sketching a Rational function $F(x)=\frac{P(x)}{Q(x)}$

1. Domain? Where $Q(x)\neq 0$
2. Asymptotes. First Cancel any common factors
3. Same-Sign Intervals. The zeros and vertical asymptotes of $F(x)$ divide the x-axis into intervals where F has the same sign.
4. Draw the asymptotes; use the same-sign interval table to sketch $F(x)$

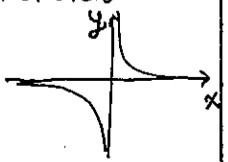
ex Sketch $f(x)=\frac{1}{x}$. "Rectangular Hyperbola"

Soln:
1. Domain: $x\neq 0$
2. Asymptotes: Vertical: $x=0$
   Horizontal: $y=0$
   Slant: none
3. Same-Sign Interval: (zeros & vertical asymptote) $f(x)$ never equals zero

x		0	
$\frac{1}{x}=f(x)$	$-$	$\pm\infty$	$+$

when $x<0$, $\frac{1}{x}<0$; when $x>0$, $\frac{1}{x}>0$

4. Sketch



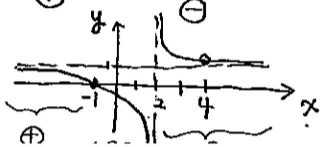
ex There is a hole in the graph when a common factor is present. Sketch $f(x)=\frac{x^2-3x-4}{x^2-6x+8}$.

Solution: $f(x)=\frac{(x-4)(x+1)}{(x-4)(x-2)}$

1. Domain: $\boxed{x\neq 4,\ x\neq 2}$
2. Asymptotes: Simplify. $f(x)=\frac{x+1}{x-2}$
   Vertical: $x=2$
   Horizontal: $y=1$
   Slant: none
3. Same-Sign Intervals? Zeros: $f(x)=0$ when $x=-1$
   Vertical asymptote $x=2$

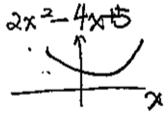
x		-1		2	
$f(x)=\frac{x+1}{x-2}$	$\frac{-}{-}$ ⊕	0	$\frac{+}{-}$ ⊖	$\pm\infty$	$\frac{+}{+}$ ⊕

4. Sketch



ex Sketch $g(x)=\frac{2x^2-4x+5}{3-x}$

Solution:
1. Domain $x\neq 3$
2. Asymptotes:
   Are there common factors? $D=4^2-4(2)(5)<0$
   The numerator is irreducible.



   Vertical: $x=3$
   Horizontal: none
   Slant: $y=-2x-2$

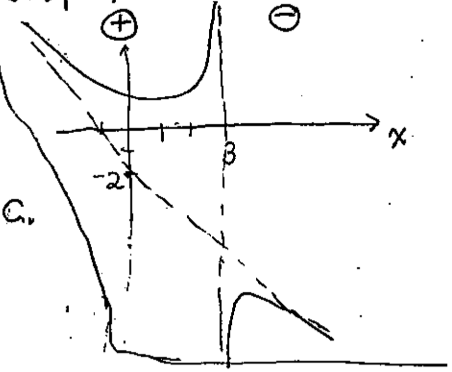
$$\begin{array}{r} -2x-2 \\ -x+3\,\overline{)\,2x^2-4x+5} \\ \underline{+2x^2-6x} \\ 2x+5 \\ \underline{2x-6} \\ 11 \end{array}$$

$f(x)=(-2x-2)+\frac{11}{3-x}$ — insignificant when $|x|\to\infty$

3. Same-Sign Intervals (No zeros. Vertical Asymptote $x=3$)

x		3	
$f(x)$	$\frac{+}{+}$ ⊕	$\pm\infty$	$\frac{+}{-}$ ⊖

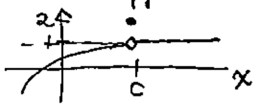
4. Sketch



(VII) Limits & Continuity

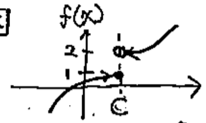
$\lim_{x\to c} f(x)$ is the value $f(x)$ approaches as x approaches but is unequal to c.

ex



$\lim_{x\to c} f(x)=1$ even though $f(1)=2$

ex



(Approach from left) $\lim_{x\to c^-} f(x)=1$
(Approach from right) $\lim_{x\to c^+} f(x)=2$

f not continuous at c

$\lim_{x\to c} f(x)$ does not exist
(For a limit to exist, $f(x)$ must approach one value no matter in what direction x approaches c)

★ f is Continuous if you can trace the graph of the function without lifting pencil from paper.

$\boxed{f \text{ is continuous at } c \text{ means } \lim_{x\to c} f(x)=f(c)}$

- Properties of limits

$$\boxed{\lim_{x\to c}\left(f(x) \pm \times \div\, g(x)\right) = \left(\lim_{x\to c} f(x)\right) \pm \times \div \left(\lim_{x\to c} g(x)\right)}$$

★ for $\lim_{x\to c}\frac{f(x)}{g(x)}$, 1st simplify. It won't work if $\frac{\lim f(x)}{\lim g(x)}=\frac{\infty}{\infty}$ or $\frac{0}{0}$

Examples of Limits

★ Note: $\lim_{x\to\infty}\frac{x^2-1}{x^2+1}=1$ $\quad\lim_{x\to-\infty}\frac{x^2-1}{x^2+1}=1$

(To find the limit as $x\to\pm\infty$, you can look at the leading terms. But this does NOT work for $x\to c$, c = finite constant.)

ex $\lim_{x\to1}\frac{x^2-1}{x+1}=\frac{1-1}{1+1}=\boxed{0}$

ex $\lim_{x\to2^+}(3x+5)$ Solution: $3(2)+5=\boxed{11}$

ex $\lim_{x\to2}\left(\frac{3x^2+5}{x-2}\right)$ Solution: $\lim_{x\to2^+}\frac{3x^2+5}{x-2}=+\infty$ $\quad\lim_{x\to2^-}\frac{3x^2+5}{x-2}=-\infty$ $\quad\lim_{x\to2}\frac{3x^2+5}{x-2}$ $\boxed{\text{does not exist}}$

Numerator is positive. Denominator >0 if $x\to2^+$, <0 if $x\to2^-$

ex $f(x)=\begin{cases}3x+2 & x\neq0\\ 0 & x=0\end{cases}$

$\lim_{x\to0}f(x)=?$ Solution $\lim_{x\to0}(3x+2)=3(0)+2=\boxed{2}$

x Approaches but is unequal to 0

ex $\lim_{x\to\infty}\frac{3x^2+4x+2}{2x^2+x-5}=?$ Solution: $\boxed{\frac{3}{2}}$

OR $\lim_{x\to\infty}\frac{3+\frac{4}{x}+\frac{2}{x^2}}{2+\frac{1}{x}-\frac{5}{x^2}}=3/2$

ex $f(x)=\frac{x^4-1}{x^3-1}$. For f to be continuous at $x=1$, what must be $f(1)$?

Solution: $f(1)=\lim_{x\to1}f(x)=\lim_{x\to1}\frac{x^4-1}{x^3-1}$

Notice $(x-1)$ is a common factor since $1^4-1=0$, $1^3-1=0$

$$\begin{array}{r} x^2+x+1\\ x-1\,\big)\,\overline{x^3-1}\\ \underline{x^3-x^2}\\ x^2-1\\ \underline{x^2-x}\\ x-1\end{array}$$

$\Rightarrow\frac{x^4-1}{x^3-1}=\frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)}=\frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{(x-1)}(x^2+x+1)}$

$f(1)=\lim_{x\to1}\frac{(x+1)(x^2+1)}{x^2+x+1}=\frac{2(2)}{1+1+1}=\boxed{\frac{4}{3}}$

OR use L'Hopital's Rule in Calculus: $\lim_{x\to1}\frac{x^4-1}{x^3-1}\overset{\frac{0}{0}}{=}\lim_{x\to1}\frac{4x^3}{3x^2}=\frac{4\cdot1^3}{3\cdot1^2}$

ex $\lim_{x\to2}\frac{x^3-8}{x^4-16}=?$ Solution: Notice $x-2$ is a common factor since $2^3-8=0$ & $2^4-16=0$.

$(P(c)=0\Leftrightarrow x-c$ is a factor of $P(x))$

$=\lim_{x\to2}\frac{(x-2)(x^2+2x+4)}{(x^2-4)(x^2+4)}$

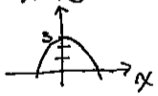
$=\lim_{x\to2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)(x^2+4)}=\frac{4+4+4}{4(4+4)}=\boxed{\frac{3}{8}}$

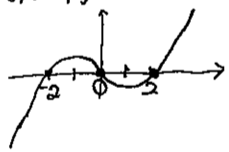
(OR by L'Hopital $\lim_{x\to2}\frac{3x^2}{4x^3}=\frac{3(4)}{4(8)}=\frac{3}{8}$)

(VIII) Piecewise-Defined Functions

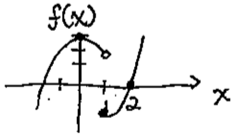
ex $f(x)=\begin{cases}3-x^2 & \text{if } x<1\\ x^3-4x & \text{if } x\geq1\end{cases}$ Sketch.

Solution: $-x^2+3$ $\qquad x(x^2-4)=x(x-2)(x+2)$



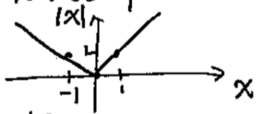


Zeros: 0, 2, −2
multiplicity: all 1 cross x-axis
{degree: 3
{leading coefficient > 0
⇒ 1 side up, left side down

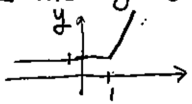


ex The absolute value can be considered piecewise-defined

$\boxed{|x|=\begin{cases}x & \text{if } x\geq0\\ -x & \text{if } x<0\end{cases}}$



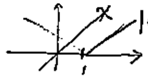
ex The figure is a graph of which?



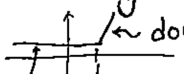
(A) $y=2x-|x|$ (B) $y=|x-1|+x$ (C) $y=|2x-1|$ (D) $y=|x+1|-x$ (E) $y=2|x|-|x|$

$\boxed{B}$ Solution: the sharp corner occurs at $x=1$. Perhaps you have an absolute value function translate right by 1 unit. (B)

Check:



$\Rightarrow$



$|x-1|+x=\begin{cases}x-1+x & \text{if } x\geq1\\ \underbrace{-(x-1)+x}_{=1} & \text{if } x<1\end{cases}$

ex Find the vertex of $y=a|bx+c|+d$.

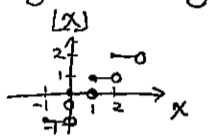
Solution: $y=a\left|b\left(x-\frac{-c}{b}\right)\right|+d$ This is a transformation of $f(x)=|x|$. $y=af\left(b\left(x-\frac{-c}{b}\right)\right)+d$

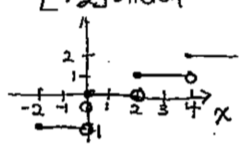
f has been stretched horizontally by b, translated to $\frac{-c}{b}$, stretched vertically by a, vertically translated to d.

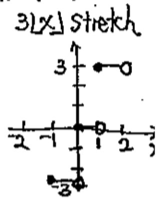
Vertex is at $\boxed{\left(-\frac{c}{b},d\right)}$

ex $\lfloor x\rfloor=\text{floor}(x)=\text{int}(x)$ = greatest integer less than or equal to x

$=\begin{cases}\vdots & \\ -1 & -1\leq x<0\\ 0 & 0\leq x<1\\ 1 & 1\leq x<2\\ \vdots & \end{cases}$



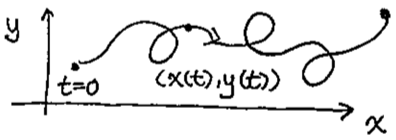




(IX) Parametric Equations

Imagine a bug flying. Its position using x and y coordinates depends on time t.

- t = time
- $(x(t), y(t))$ location in plane varies with time



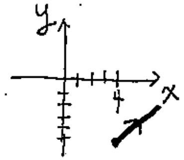
(Maybe y is not a function of x
Maybe x is not a function of y)

1. Eliminate the parameter t to sketch y versus x. (Maybe use ID $\sin^2 t+\cos^2 t=1$)
2. Be careful with restrictions.

ex Sketch $\begin{cases} x=3t+4 \\ y=t-5 \end{cases}$, $t\geq 0$

Solution: 1. Eliminate t

$t=\frac{x-4}{3} \Rightarrow y=\frac{x-4}{3}-5 = \frac{1}{3}x-\frac{19}{3}$



2. Restriction: $t\geq 0$

$x\uparrow$ as $t\uparrow \Rightarrow x\geq 3(0)+4=4$

$y\uparrow$ as $t\uparrow \Rightarrow y\geq 0-5=-5$

ex Find the slope of the line represented by parametric equations $\begin{cases} x=3t+4 \\ y=t-5 \end{cases}$.

solution: The fastest way is to pick to convenient points and find the rise/run.

$t=0$: $(x(0),y(0))=(4,-5)$
$t=1$: $(x(1),y(1))=(7,-4)$
$\Rightarrow$ slop $=\frac{-4+5}{7-4}=\boxed{\frac{1}{3}}$

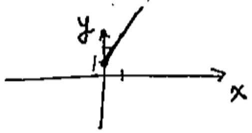
ex Sketch $\begin{cases} x=t^2 \\ y=3t^2+1 \end{cases}$ $t\in\mathbb{R}$.

solution: 1. Eliminate t.

$y=3x+1$

2. Restriction: $x=t^2\geq 0$, $y=3t^2+1\geq 1$

since $t\uparrow \Rightarrow x\uparrow$ & $y\uparrow$



ex Sketch $\begin{cases} x=4\cos\theta \\ y=3\sin\theta \end{cases}$

Solution: 1. Eliminate θ.

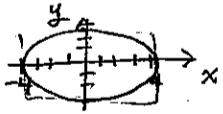
$\sin^2\theta+\cos^2\theta=1$

$\boxed{\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2=1}$ Ellipse

2. Restrictions

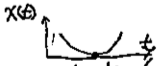
$-1\leq\sin\theta\leq 1 \Rightarrow \boxed{-3\leq y\leq 3}$

$-1\leq\cos\theta\leq 1 \Rightarrow \boxed{-4\leq x\leq 4}$



ex $\begin{cases} x=t^2+t \\ y=t^2-t \end{cases}$ What are the restrictions on x and y?

Solution: $x(t)=t^2+t$, (at^2+bt+c)

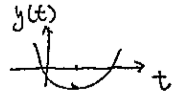


Vertex $\left(-\frac{b}{2a}, x\left(-\frac{b}{2a}\right)\right)=\left(-\frac{1}{2}, \left(\frac{1}{2}\right)^2+\frac{1}{2}\right)$
$=\left(-\frac{1}{2},-\frac{1}{4}\right)$

$\boxed{x(t)\geq -\frac{1}{4}}$

$y(t)=t^2-t$ vertex is at $\left(\frac{1}{2},\left(\frac{1}{2}\right)^2-\frac{1}{2}\right)=\left(\frac{1}{2},-\frac{1}{4}\right)$

$\boxed{y(t)\geq -\frac{1}{4}}$



ex $\begin{cases} x=\sin^2 t \\ y=2\cos t \end{cases}$ $t\in\mathbb{R}$. Sketch y versus x.

Solution: 1. Eliminate t.

$\sin^2 t+\cos^2 t=1$

$x+\left(\frac{y}{2}\right)^2=1$

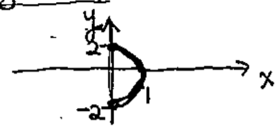
$y^2=-4(x-1)$

Parabola opening left, Vertex at $(1,0)$

($(y-k)^2=4p(x-h)$ Vertex (h,k)
$p=-1<0$
left)

2. Restrictions

$\boxed{0\leq x\leq 1}$
$\boxed{-2\leq y\leq 2}$



Systems of Equations & Inequalities

I. Rules: a, b, c are all real numbers

(1) $a=b$ is equivalent to the following equations:

$a+c=b+c$ for any c

★ $\frac{a}{c}=\frac{b}{c}$ only for $c\neq 0$

$ac=bc$ only for $c\neq 0$

(2) $|a|=\begin{cases} a & \text{if } a\geq 0 \\ -a & \text{if } a<0\end{cases}$ = distance from origin

★ So $|x-y|=|-(x-y)|=|y-x|$

(3) $\sqrt[n]{a^n}=\begin{cases} |a| & \text{if } n \text{ is an even integer} \\ a & \text{if } n \text{ is an odd integer}\end{cases}$ $\quad \sqrt{(-2)^2}=2$, $\sqrt[3]{(-2)^3}=-2$

(4) For $a, b\geq 0$

$a\leq b \Rightarrow \sqrt{a}\leq\sqrt{b}$

ex. ★ $x^2\leq 1$, $\sqrt{x^2}\leq\sqrt{1}$, $|x|\leq 1$, $-1\leq x\leq 1$

(5) If $a\leq b$

then $-a\geq -b$, $\frac{1}{a}\geq\frac{1}{b}$, $-\frac{1}{a}\leq -\frac{1}{b}$

$ca\overset{\text{same}}{\leq} cb$ if $c>0$

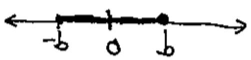
★ $ca\underset{\text{flip}}{\geq} cb$ if $c>0$

(6) Be careful of taking powers in an inequality

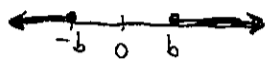
If $a\leq b$
then $a^3\leq b^3$ for any $a, b\in\mathbb{R}$
but $a^2\leq b^2$ only if $|a|\leq|b|$

ex. $-7<2$
$(-7)^2>2^2$
$(-7)^3<2^3$

(7) $|a|\leq b$ means $-b\leq a\leq b$



$|a|\geq b$ means $a\leq -b$ or $a\geq b$

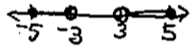


(8) $|f(t)|=g(t)$ means $f(t)=g(t)$ or $f(t)=-g(t)$ with restriction $g(t)\geq 0$. Check for extraneous solutions!

(9) $AB\geq 0$ means $(A\geq 0 \ \& \ B\geq 0)$ or $(A\leq 0 \ \& \ B\leq 0)$
$AB\leq 0$ means $(A\geq 0 \ \& \ B\leq 0)$ or $(A\leq 0 \ \& \ B\geq 0)$

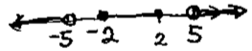
★ Inequalities in 1 variable: Solution is an interval

- ex Solve $|x|>3$ and $|x|\leq 5$ ($\cap$)



Soln: $\boxed{[-5,-3)\cup(3,5]}$

- ex Solve $|x|\leq 2$ or $|x|>5$ ($\cup$)



Soln: $(-\infty,-5)\cup[-2,2]\cup(5,\infty)$

ex Solve $-2|x+1|+5\geq -3$

solution: 1. Isolate absolute Value
$5+3\geq 2|x+1|$
$8\geq 2|x+1|$
$|x+1|\leq 4$

2. Write Compound inequality
$-4\leq x+1\leq 4$
$-5\leq x\leq 3$

- ex Solve $5\leq 7-x\leq 8$

Solution $-2\leq -x\leq +1$
$2\geq x\geq -1$

- ex Solve $|2x+5|-4=3x$

Solution: (1) Isolate the absolute Value
$|2x+5|=3x+4$

(2) Eliminate absolute value & solve.

★ Restriction: $3x+4\geq 0$
$x\geq -4/3$

$2x+5=3x+4$ or $2x+5=-3x-4$

$1=x$ | $5x=-9$, $x=-9/5$

Check: $x=1\geq -4/3$? ✓ | Check: $-9/5\geq -4/3$? No!

Solution is $\boxed{x=1}$

"extraneous solution" (NOT a true solution)

ex Solve

$|-x^2-1|=5$

Solution: $|-x^2-1|=|\underbrace{x^2+1}_{\geq 0}|=x^2+1 \Rightarrow x^2+1=5$, $x^2=4$ $\boxed{x=2 \text{ or } -2}$

- ex Solve $|x-3|=-2$

Solution: No solution since an absolute value cannot be negative

ex Write without absolute value symbols:

$\left|\frac{x+7}{|x|+|x-1|}\right|$, given $0<x<1$. Solution: $\frac{|\overset{\geq 0}{x+7}|}{\underset{\geq 0}{|x|}+\underset{<0}{|x-1|}}=\frac{x+7}{x-(x-1)}=\boxed{x+7}$

ex Solve $2+\frac{5x}{x-6}=\frac{30}{x-6}$.

Solution: Method 1: $2+\frac{5x-30}{x-6}=0$

$2+\frac{5(x-6)}{x-6}=0$

$7=0$. This is an inconsistent equation. No x could make it true. $\boxed{\text{No solution}}$

Method 2:

$(x-6)\left(2+\frac{5x}{x-6}\right)=\frac{30}{x-6}(x-6)$

$\boxed{x-6\neq 0,\ x\neq 6}$ Only okay when you multiply by a nonzero number on both sides.

$2x-12+5x=30$

$7x=42$

$x=6$ does not satisfy restriction $x\neq 6$

No solution.

ex Solve $-5(4-x)<5x$

Soln: $-20+5x<5x$

$-20<0$ Always true. x can be any real number.

Solution is $\boxed{\text{the set of all real numbers}}$

ex Solve $x^2+3x-4<0$

Solution: $(x-4)(x+1)<0$ $\quad$ -1 $\quad$ 4 $\quad$ $\boxed{-1\leq x<4}$

ex Solve $\frac{3x+4}{x+1}\leq 2$ $\quad$ $x\neq -1$

Solution:

Method 1: • Multiply both sides by a positive number to preserve the inequality

$(x+1)^2\frac{(3x+4)}{x+1}\leq 2(x+1)^2$

$(x+1)(3x+4)\leq 2(x^2+2x+1)$

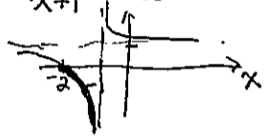
$3x^2+7x+4\leq 2x^2+4x+2$

$x^2+3x+2\leq 0$

$(x+1)(x+2)\leq 0$ $\quad$ -2 $\quad$ -1

$\boxed{-2\leq x<-1}$

• Check by graphing calculator $\frac{3x+4}{x+1}-2$



Method 2: When $x+1>0$ ($x>-1$)

$(x+1)\left(\frac{3x+4}{x+1}\right)\leq 2(x+1)$

$3x+4\leq 2x+2$

$x\leq -2$

But $x\leq -2$ & $x>-1$ is impossible

when $x+1<0$ ($x<-1$)

$(x+1)\left(\frac{3x+4}{x+1}\right)\geq 2(x+1)$

$3x+4\geq 2x+2$

$x\geq -2$

$\boxed{-2\leq x<-1}$ OK

solution

Systems of Equations

• in 2 variables, the solution is the intersection of curves

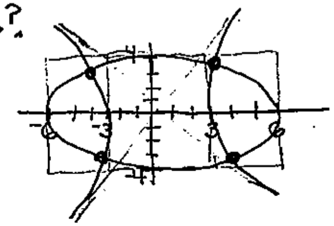
• in 3 variables, the solution is the intersection of planes

• ex A nonlinear system of equations.

How many solutions are there?

$$\begin{cases}\frac{x^2}{36}+\frac{y^2}{16}=1 & \text{ellipse}\\ \frac{x^2}{9}-\frac{y^2}{16}=1 & \text{hyperbola}\end{cases}$$

Solution:



$\boxed{\text{4 points in solution set}}$

III Matrices & Linear System of Equations

ex Scalar Multiplication & Addition

$$3\begin{bmatrix}-2&3\\1&5\\-4&3\end{bmatrix}-2\begin{bmatrix}3&-1\\2&1\\-4&6\end{bmatrix}=\begin{bmatrix}-6&9\\3&15\\-12&9\end{bmatrix}-\begin{bmatrix}6&-2\\4&2\\-8&12\end{bmatrix}=\begin{bmatrix}-12&11\\-1&13\\-4&-3\end{bmatrix}$$

ex Multiplication of matrices. In general $AB\neq BA$

$\underset{n\times r\ \ r\times m}{AB}$ has dimensions $n\times m$

$$C=\underbrace{\begin{bmatrix}3&-1\\3&5\\-2&1\end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix}1&-2\\5&-3\end{bmatrix}}_{2\times 2}=\begin{bmatrix}3\cdot1-1\cdot5=-2 & 3\cdot-2+-1\cdot-2=-4\\ 3\cdot1+5\cdot5=28 & 3\cdot-2+5\cdot-3=-21\\ -2\cdot1+1\cdot5=3 & -2\cdot-2+1\cdot-3=1\end{bmatrix}=\underbrace{\begin{bmatrix}-2&-4\\28&-21\\3&1\end{bmatrix}}_{3\times 2}$$

$c_{ij}=(i^{th}\text{ row of }A)\times(j^{th}\text{ row of }B)$

$c_{21}=[3\ \ 5]\begin{bmatrix}1\\5\end{bmatrix}=3\cdot1+5\cdot5=28$

ex Determinant $\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$

Determinant of a matrix

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ ex $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \\ 4 & 1 & 6 \end{vmatrix} = ?$

Solution: $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \\ 4 & 1 & 6 \end{vmatrix} = -3 \begin{vmatrix} -1 & 4 \\ 1 & 6 \end{vmatrix} + 0 - 5 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$

$= -3(-6-4) + 0 - 5(2+4) = +30 - 30 = \boxed{0}$

OR $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \\ 4 & 1 & 6 \end{vmatrix} = --1 \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 0 - 1 \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \cdots$

• Inverse of a square matrix A $\boxed{A^{-1} \text{ exists} \Leftrightarrow |A| \neq 0}$

$A^{-1} \cdot A = A \cdot A^{-1} = I$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

★ Matrices in Linear Systems of Equations — highest powers are 1: $Ax+By+Cz=D$

- Same number of unknowns as equations
- Coefficient matrix C is square

$\begin{cases} |C| \neq 0 \Rightarrow \text{unique solution} \\ |C| = 0 \Rightarrow \text{infinitely many or no solution. Solve algebraically.} \end{cases}$

ex
$$\begin{aligned} x - y + 2z &= -3 \\ y + 2x - z &= 0 \\ -x + 2y - 3z &= 7 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

C coefficient matrix

$|C| \neq 0 \Leftrightarrow C^{-1}$ exists

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1} \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}}$$

use graphing calculator

• ex Find $y = ax^2 + bx + c$ through points $(1,4), (-1,6), (2,9)$

Solution:
$$\begin{aligned} a(1)^2 + b(1) + c &= 4 \\ a(-1)^2 + b(-1) + c &= 6 \\ a(2)^2 + b(2) + c &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}$$

calculator: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

★ If there are fewer equations than unknowns, then there are infinitely many or no solution.

ex Solve $\begin{cases} 2x - y - 2z = 15 \\ x - 2y + 2z = 3 \end{cases}$

Solution: Get x and z in terms of y.

• $2x - y - 2z = 15$
$x - 2y + 2z = 3$
$3x - 3y = 18$
$x = y + 6$

• Get z in terms of y $\rightarrow$ $2z = 3 - x + 2y$
$= 3 - y - 6 + 2y$
$= y - 3$
$z = \frac{y-3}{2}$

Infinitely many Solutions: $\boxed{\{(b+6, b, \frac{b-3}{2}) : b \in \mathbb{R}\}}$

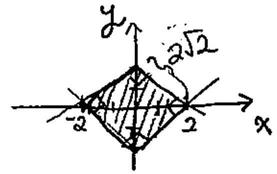
ex Find the area bounded by $|x| + |y| = 2$

Solution: Try to sketch the curve.

$|y| = -|x| + 2$

• $y = -|x| + 2$ or $y = |x| - 2$

• Restriction: $|y| = 2 - |x|$ so $2 - |x| \geq 0$
$|x| \leq 2$
$-2 \leq x \leq 2$



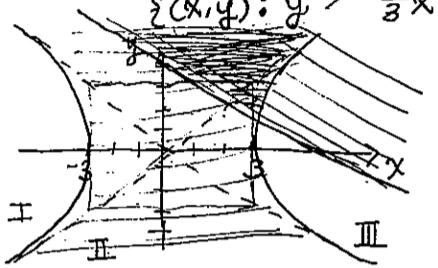
square $(2\sqrt{2})^2 = \boxed{8}$

III System of Inequalities

★ for a system of inequalities in 2 variables, the solution set is the intersection of regions.

• ex Graph the solution set of $\begin{cases} x^2 - y^2 \leq 9 & ② \\ 2x + 3y > 12 & ① \end{cases}$

Soln: ① means $\{(x,y) : y > -\frac{2}{3}x + 4\}$



The solution is the dark region

② $(\frac{x}{3})^2 - (\frac{y}{3})^2 = 1$ is a hyperbola which divides the x,y plane into regions I, II, III. Test which regions fit the inequality.

I. $(-4,0)$ $(-4)^2 - 0^2 \leq 9$? No

II. $(0,0)$ $0^2 - 0^2 \leq 9$? Yes

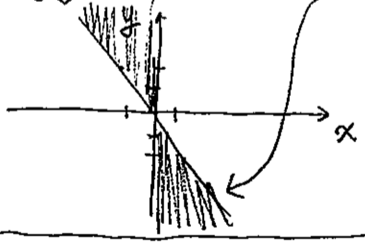
III. $(4,0)$ $4^2 - 0^2 \leq 9$? No

Region II is the set $\{(x,y) : x^2 - y^2 \leq 9\}$

ex Graph $x(y+2x) \le 0$

Solution: $x \le 0$ and $y+2x \ge 0$ OR $x \ge 0$ and $y+2x \le 0$

$x \le 0$ & $y \ge -2x$ OR $x \ge 0$ & $y \le -2x$

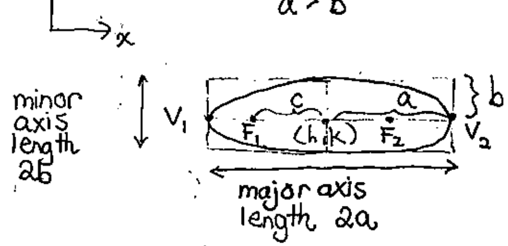


Coordinate Geometry

I. Conic Sections

(1) Ellipse $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$

$a > b$



Center: (h,k)

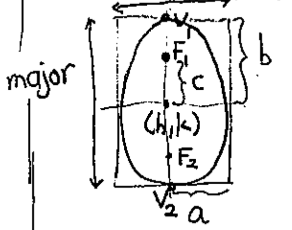
Vertices: $V_1(h-a,k)$, $V_2(h+a,k)$ (along major axis)

Foci: $F_1(h-c,k)$, $F_2(h+c,k)$ (along major axis)

$C = \sqrt{a^2-b^2}$ ← the bigger one

eccentricity $e = \frac{c}{a}$ ← the bigger one

$a < b$



center (h,k)

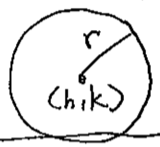
Vertices: $V_1(h,k+b)$, $V_2(h,k-b)$

Foci: $F_1(h,k+c)$, $F_2(h,k-c)$

$c = \sqrt{b^2-a^2}$

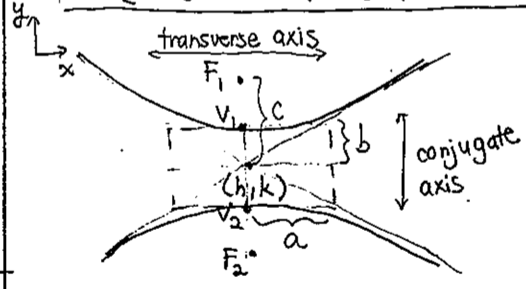
$e = c/b$

Circle: $(x-h)^2 + (y-k)^2 = r^2$



(2) Hyperbola

$-\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$



center (h,k)

Vertices: $(h,k+b)$, $(h,k-b)$

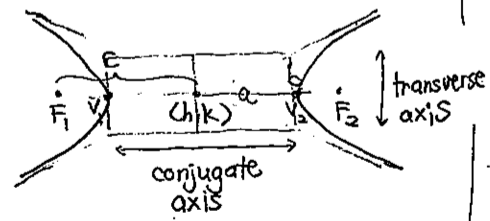
Foci: $F_1(h,k+c)$, $F_2(h,k-c)$

$\boxed{c = \sqrt{a^2+b^2}}$ $e = \frac{c}{b}$

Asymptotes: $\boxed{\pm\frac{b}{a} = \frac{y-k}{x-h}}$

$y = \pm\frac{b}{a}(x-h)+k$

$\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1$



center (h,k)

Vertices: $(h-a,k)$, $(h+a,k)$

Foci: $(h-c,k)$, $(h+c,k)$

$c = \sqrt{a^2+b^2}$

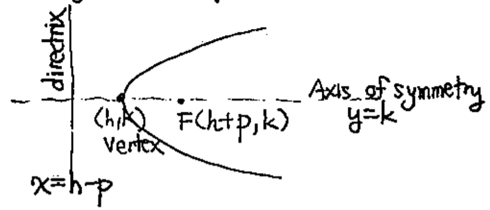
Asymptotes: $\pm\frac{b}{a} = \frac{y-k}{x-h}$

eccentricity $e = \frac{c}{a}$

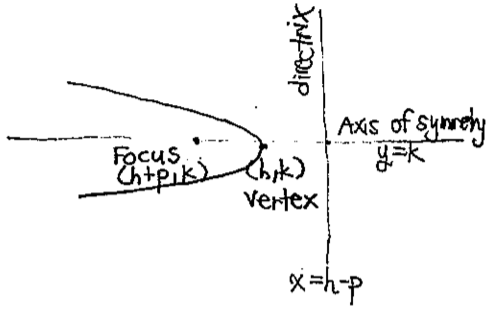
(3) Parabola

$(y-k)^2 = 4p(x-h)$

$p > 0$

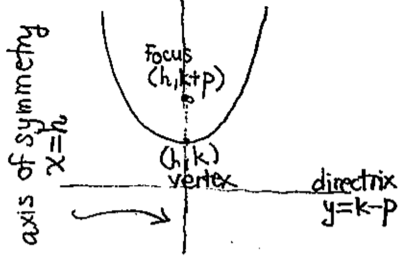


$p < 0$

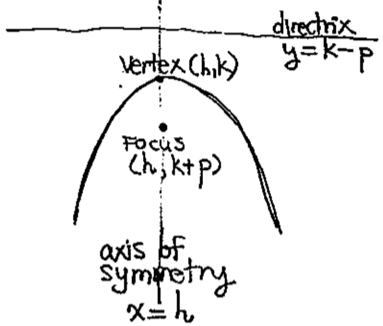


$(x-h)^2 = 4p(y-k)$

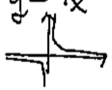
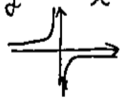
$p > 0$



$p < 0$



- **Conic Sections** (Ellipse, Circle, Hyperbola, Parabola) are curves described by the general quadratic equation. $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$
- On the SAT 2 Math Level 2, Usually $B=0$ (which handles rotations). The exception is the rectangular hyperbola. $xy=k$ $\quad y=\frac{1}{x}$ $\quad y=-\frac{1}{x}$

- To sketch the graph,
  $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$
  (1) Complete the square to get standard form of $\begin{cases}\text{parabola}\\ \text{ellipse}\\ \text{hyperbola}\end{cases}$
  (2) Identify the conic section

	Could be
$Ax^2+Cy^2+Dx+Ey+F=0$	Ellipse Hyperbola
$Ax^2+Dx+Ey+F=0$	Parabola opening up or down
$Cy^2+Dx+Ey+F=0$	Parabola opening left or right

ex Identify, Sketch, Label

$9x^2-16y^2-18x+96y+9=0$

Solution: x^2 & $y^2 \Rightarrow$ ellipse or hyperbola. Complete the square for x & for y.

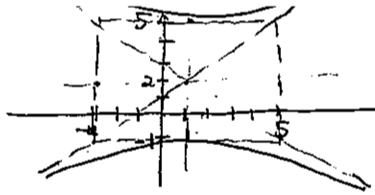
$$9(x^2-\underbrace{2x}_{2(1)}+(1)^2)-16(y^2-\underbrace{6y}_{2(3)}+3^2)=-9+9(1)^2-16(3)^2$$

$$9(x-1)^2-16(y-3)^2=-144$$

$$-\frac{(x-1)^2}{16}+\frac{(y-3)^2}{9}=1$$

$$\boxed{-\left(\frac{x-1}{4}\right)^2+\left(\frac{y-3}{3}\right)^2=1}$$

hyperbola opens in y-direction



center $(1,3)$
Vertices: $(1,5),(1,-1)$
Foci: $c=\sqrt{4^2+3^2}=5$
$(1,3+5),(1,3-5)$
eccentricity $e=\frac{c}{3}=\frac{5}{3}$

Asymptotes: $\pm\frac{3}{4}=\frac{y-3}{x-1}$

$$\boxed{y=\frac{3}{4}(x-1)+3 \quad \& \quad y=-\frac{3}{4}(x-1)+3}$$

ex Identify, Sketch, Label

$y^2+6x-8y+4=0.$

Solution: x^1, y^2 parabola left or right

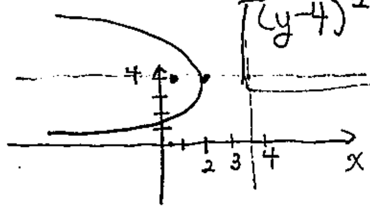
Complete the square for y

$$(y^2-\underbrace{8y}_{2(4)}+(4)^2)+6x+4=0+4^2$$

$$(y-4)^2=-6x+12$$

$$\boxed{(y-4)^2=-6(x-2)}$$

$4p$, $p=-\frac{3}{2}<0$ opens left



vertex: $(2,4)$
Focus: $(2-\frac{3}{2},4)$
directrix: $x=2+3/2$
axis of symmetry: $y=4$

ex An equilateral triangle is inscribed in the circle $x^2+2x+y^2-4y=0$. What is the length of each side of the triangle?

Solution: Circle $(x^2+2(1)x+(1)^2)+y^2-2(2)y+(2)^2=1^2+2^2$

$(x+1)^2+(y-2)^2=5$

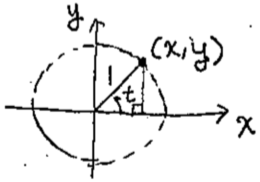
Radius $\sqrt{5}$

$\tan 60^\circ=\frac{l}{r}$

$\Rightarrow l=\sqrt{5}\tan 60^\circ \Rightarrow 2l=\boxed{3.87}$

(II) Trigonometry

1. Unit Circle : $\sin t \equiv$ y-coordinate on unit circle (when the terminal side of length 1 is rotated by an angle of t from the positive x-axis).



$\sin t \equiv y$ $\qquad \csc t = \frac{1}{y} = \frac{1}{\sin t}$

$\cos t = x$ $\qquad \sec t = \frac{1}{x} = \frac{1}{\cos t}$

$\tan t = \frac{y}{x} = \frac{\sin t}{\cos t}$ $\qquad \cot t = \frac{x}{y} = \frac{\cos t}{\sin t} = \frac{1}{\tan t}$

2. All Students Take Calculus

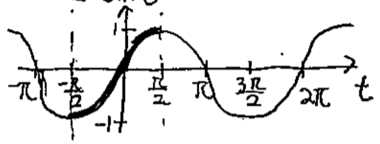
- Quadrant II: only (s)int ≥ 0, csct
- Quadrant I: All trig functs ≥ 0
- Quadrant III: only (t)ant ≥ 0, cott ≥ 0
- Quadrant IV: Only (c)ost ≥ 0, sect ≥ 0

3.

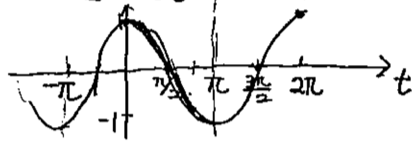
	Domain	Range	Symmetry
$\sin t$	$\mathbb{R}$	$-1 \leq \sin t \leq 1$	odd function
$\cos t$	$\mathbb{R}$	$-1 \leq \cos t \leq 1$	even function
$\tan t$	$t \neq \frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$	$\mathbb{R}$	odd
$\csc t$	$t \neq k\pi, k \in \mathbb{Z}$	$-1 \leq \sin t \leq 1$; $\frac{1}{\sin t} \leq -1, \frac{1}{\sin t} \geq 1$	odd
$\sec t$	$t \neq \pm\frac{3\pi}{2}, \pm\frac{\pi}{2}, \ldots$	$\frac{1}{\cos t} \leq -1, \frac{1}{\cos t} \geq 1$	even
$\cot t$	$t \neq k\pi$	$\mathbb{R}$	odd

4. Trigonometric Inverses

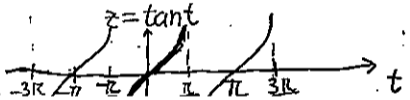
$z = \sin t$



$z = \cos t$



$z = \tan t$



$\sin^{-1} z : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$z \longmapsto t$ (angle s.t. $\sin t = z$)

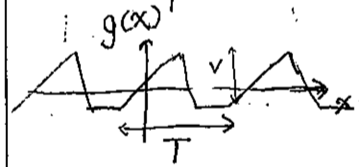
$\cos^{-1} z : [-1,1] \longrightarrow [0,\pi]$

$z \longmapsto t$ (angle s.t. $\cos t = z$)

$\tan^{-1} z : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$z \longmapsto t$ (angle s.t. $\tan t = z$)

- For a periodic function $g(x)$



- Period T = smallest number such that $g(x+T) = g(x)$
- Amplitude = $\frac{1}{2}V$ vertical distance between maximum & minimum

ex If $g(x)$ has period T,

$y = \underbrace{Dg(Ax+B)+C}_{h(x)}$ has period $\frac{T}{A} = \boxed{\frac{\text{original period}}{\text{constant in front of } x}}$

Find P $\quad h(x+P) = h(x)$

$\cancel{D}g(Ax + AP + B) \cancel{+C} = \cancel{D}g(Ax+B) + \cancel{C}$

$\Rightarrow AP = T \Rightarrow P = \frac{T}{A}$

- For a Sinusoid, $f(x) = \sin x$ or $\cos x$

If $y = Af(Bx+C)+D$, then the graph has period $\boxed{\frac{2\pi}{B}}$

$= Af\left(B\left(x + \frac{C}{B}\right)\right) + D$

- Horizontal Translation = Phase Shift $\boxed{-\frac{C}{B}}$
- Amplitude : $\boxed{A}$

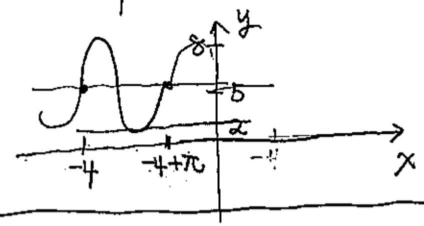
ex sketch $y = 3\sin(2x+8)+5$. Amplitude? Period? Phase Shift?

Soln: $y = 3\sin(2(x+4))+5$

Amplitude: 3

Period: $\frac{2\pi}{2} = \pi$

Phase Shift: -4 (4 left)



Identities

$\boxed{\begin{array}{l} \sin^2 t + \cos^2 t = 1 \\ \sec^2 t = 1 + \tan^2 t \\ \csc^2 t = 1 + \cot^2 t \end{array}}$

When $\alpha + \beta = 90°$

$\sin\alpha = \cos\beta$

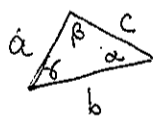
$\sec\alpha = \csc\beta$

$\tan\alpha = \cot\beta$

$\boxed{\begin{array}{l} \sin x = \cos(\frac{\pi}{2} - x) \\ \sec x = \csc(\frac{\pi}{2} - x) \\ \csc \qquad \sec \\ \tan x = \cot(\frac{\pi}{2} - x) \\ \sin(-x) = \cos(\frac{\pi}{2} - x) \end{array}}$

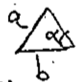
$\boxed{\begin{array}{l} \sin 2x = 2\sin x \cos x \\ \cos 2x = \cos^2 x - \sin^2 x \\ \quad = 1 - 2\sin^2 x \\ \quad = 2\cos^2 x - 1 \end{array}}$

- Law of Sines



$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

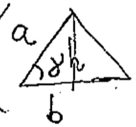
Useful when 2 sides & ∠ opposite



2 angles & side opposite

- Law of Cosines (2 sides & ∠ between) #45 M1

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

area $= \frac{1}{2}ab\cos\gamma$ ( $\frac{1}{2}b(h)$, $\frac{h}{a} = \cos\gamma$)

- series

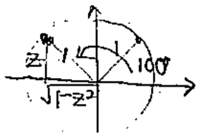
$$1+2+\ldots+N = \frac{N(N+1)}{2}$$

M1 #24

$$r^0 + r^1 + r^2 + \ldots + r^N = \frac{1-r^{N+1}}{1-r}$$

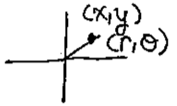
- Trig.

$\boxed{ex}$ $\sin 100° = z$. $\sin 200°$ in terms of z?



$$\sin 200° = 2\sin 100° \cos 100°$$
$$= 2z\left(-\sqrt{1-z^2}\right)$$

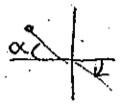
Polar



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\begin{cases} r = \sqrt{x^2+y^2} \\ \alpha = \tan^{-1}\left(\frac{y}{x}\right) = \text{reference angle of } \theta \end{cases}$

careful of Quadrant

$\boxed{ex}$ $(-1, 8)$ to polar coordinates



$$r = \sqrt{(-1)^2 + 8^2} = \sqrt{65}$$
$$\alpha = \tan^{-1}\left(\frac{8}{-1}\right) = -82°$$
$$\theta = 180° - 82° = 97.13°$$

Diag Test

Geo. 3, 15, 20, 31 ~ 33, 43, 49

Asymptotes 11 ~ 12, 28

Functions 19, 37, 45, 46

Lines 13,

Seq, Series, Stats 6, 16, 18, 34, 35

Algebra 9, 10

Log 47

Graph 23, 41, 50, 40

Review Handout for Algebra 2

- **Functions in General:**

HW2 #2, 3, 6, 7, 28, 29, 32, 43~46
HW3 #33
HW5 #98~101, 102

(Pg.1) I. Definition
(Pg.2) II. Inverse
(Pg.3) III. Domain, Range
(Pg.3) IV. Composition
(Pg.4) V. Symmetry
(Pg.5) VI. Periodic Functions
(Pg.5) VII. Transformations

- **Special Functions**

(Pg.7) I. Exponential, Logarithmic — HW2 # 11~17, 19, 25; HW3 #15
(Pg.9) II. Linear — HW2 #21, 33
(Pg.9) III. Quadratic — HW2 #27. HW3 #12, 13, 24
(Pg.11) V. Higher-Order Polynomials — HW3 #7, 8, 10, 11, 17, 31; HW5 #111~113
(Pg.14) VI. Rational — HW3 #16, 22, 23, 37
(Pg.17) VII. Piecewise — HW5 #59, 62
(Pg.18) VIII. Parametric — HW5 #51~56

- **Systems of Equations & Inequalities**

(Pg.19) I. Rules: 1-variable inequalities & equalities
(Pg.10) • Shortcuts
(Pg 20) II. Matrices & Linear Systems
(Pg.21) III. Systems of Inequalities

HW5 #66~74, 76~91~ 1, 95, 96, 100

- **Coordinate Geometry:**

HW1 #6~9, 13~17
HW3 #25
HW5 #5, 6 9~13, 23~26, 33~35
41, 42, 44~48

(Pg.22) I. Conic Sections
(Pg.24) II. Trigonometry (Lesson #7 Handout)

HW6 #1~15, 18, 19, 22~27, 30~32
HW7 #3~8, 11~15, 33, 41, 42

- Sequences & Series (Lesson #8)
  Counting: Permutations, Combinations

HW8 #2, 4, 6~20, 32

- Probability (Lesson #9)

HW9 #1~8

- Complex numbers

cycles of 4
$$\begin{cases} i = \sqrt{-1} \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{cases}$$

$(i^{89} = i^{4\times 11+1} = i^1 = i)$

HW5 # 103