

Book: Barron's SAT Subject Test Math Level 2
2008 8th Edition
by Richard Ku and Howard P. Dodge

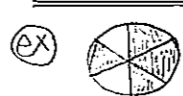
Schedule HW 2,3,5,6,7

- 8/31, 9/7 Lessons #8, #9
(Miscellaneous 10%)
Sequences, Series, Probability, Statistics
Word Problems
- 9/14, 9/21, 9/28 (Major Topics)
 - Absolute Value, System of Equations / Inequalities
 - Functions in General
 - Definition
 - Inverse
 - Transformations
 - Symmetry
 - Period
 - Specific Functions
 - Log, Exponents
 - Polynomials { Linear, Quadratic, Parabola Fundamental Thm., Higher-Order, Roots, Descartes,
 - Rational, Asymptotes, Limits
 - Conic Sections, complete the square (General quadratic)
 - Trigonometry: Right triangles, unit circle, polar coordinates, IDs
 - Geometry
 - Piecewise, parametric
 - Complex Numbers, Vectors
- 5 Practice Tests: 10/5, 10/12, 10/19, 10/26, 11/2

Lesson #9 Handout (9/7/2009)

- | | |
|----------------|-------------------|
| I. Probability | III. Statistics |
| II. Appendix | IV. Word Problems |

I. Probability



(ex) A cake with 6 equal slices.
4 are chocolate, 2 are vanilla.

If you close your eyes and take one slice, the probability of getting chocolate is $\frac{4}{6} = \frac{2}{3}$. This means if you do the experiment many times, you generally get a chocolate slice two out of every three tries. If the cake were all chocolate, you would be certain to get a chocolate slice even with your eyes closed (probability = 1), and it would be impossible to get a banana slice (probability = 0).

(S, E, P)

Sample Space S = set of all outcomes of an experiment

Event A = subset of the sample space.
We say an event occurs if any outcome in A occurs in the experiment.

Probability $P(A)$ = a function that maps an event to a number in $[0, 1]$

Event Space $\mathcal{E}$ = class of all possible events
= set of all subsets of the sample space

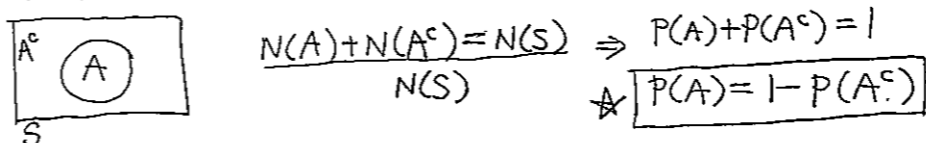
($\mathcal{E}$ is a power set of S
If S has n outcomes, $\mathcal{E}$ has 2^n sets)

$P: \mathcal{E} \rightarrow [0, 1]$
 $A \mapsto P(A)$

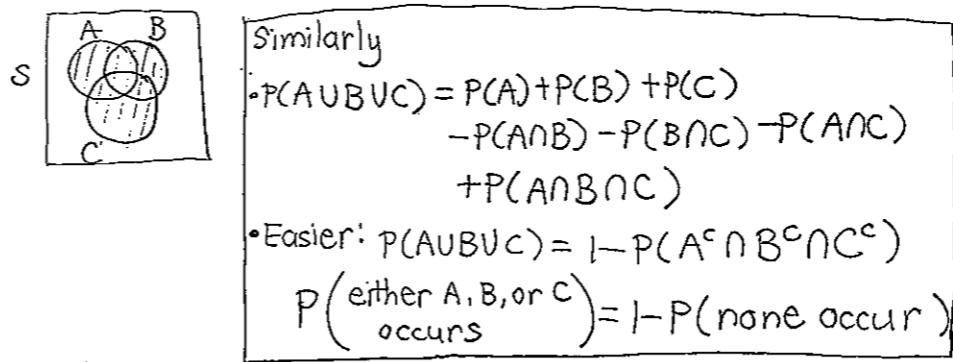
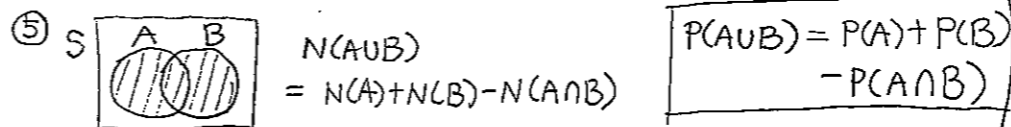
① Probability that event A occurs: $P(A) = \frac{N(A)}{N(S)}$
 $= \frac{\# \text{ of elements in } A}{\# \text{ of elements in Sample Space}}$

② $0 \leq P(A) \leq 1$ for any $A \in \mathcal{E}$

③ A event. A^c = event that A does not occur
 $A \cup A^c = S$



④ A or B occurs $\leftrightarrow A \cup B$
 A and B both occur $\leftrightarrow A \cap B$
 $P(\text{at least one}) = 1 - P(\text{none})$



⑥ Independent Events: Events A and B are

independent iff $P(A \cap B) = P(A)P(B)$

- When A occurs, it has no effect on whether B occurs
- Example: Flipping a coin once does not affect future flips.

⑦ Mutually Exclusive Events

A and B are mutually exclusive if these events cannot ever occur together.

A and B are mutually exclusive iff $P(A \cap B) = 0$

(1) which implies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $= P(A) + P(B)$

Example: It cannot be raining and dry outside at the same time.

(2) Mutually Exclusive Events are dependent

• Not independent since $P(A \cap B) = 0$

$P(A) > 0, P(B) > 0 \Rightarrow P(A)P(B) \neq 0$

$P(A \cap B) \neq P(A)P(B)$

• Intuitively, mutually exclusive events are dependent since if one occurs, the other cannot occur.

Examples

⑧ what is the probability of getting a head when a fair coin is flipped?

Soln: $S = \{H, T\}$
 $A = \{H\}$
 $P(A) = \frac{N(A)}{N(S)} = \frac{1}{2}$

⑨ what is the probability of getting a sum of 7 when two dice are thrown?

Soln: $S = \text{all possible outcomes from throwing two dice}$
 $= \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4), \dots \\ (2,1), (3,1), (4,1) \\ (2,2), (2,3), \dots \\ (3,2) \end{matrix} \right\}$
 $N(S) = \frac{6 \times 6}{1 \times 6 \times 1 \times 6} = 36$

$A = \left\{ \begin{matrix} (1,6), (2,5), (3,4) \\ (6,1), (5,2), (4,3) \end{matrix} \right\}$
 $N(A) = 6$

$P(A) = \frac{N(A)}{N(S)} = \frac{6}{36} = \frac{1}{6}$

[ex3] (A) Probability of getting a 7 when one die is thrown?

o impossible. A die is labeled 1 to 6

(B) Probability of getting a number less than 12 when one die is thrown = 1

[ex4] What are the odds in favor of getting a number greater than 2 when one die is thrown?

Soln: $S = \{1, 2, 3, 4, 5, 6\}$

$A =$ event of a number greater than 2
 $= \{3, 4, 5, 6\}$

$A^c = \{1, 2\} =$ event of getting a number ≤ 2

"Odds in favor of A" means $\frac{P(A)}{P(A^c)} = \frac{N(A)/N(S)}{N(A^c)/N(S)} = \frac{4}{2}$

[2:1]

[ex5] (Independent Events)

If two coins are flipped, what's the probability of getting two heads?

Soln: Method 1 $S = \{H, T\}$
 $A =$ event of a head on 1st coin
 $B =$ event of a head on 2nd coin

A and B are independent

$$P(A \cap B) = P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Method 2 $S = \{HH, HT, TH, TT\}$

$E =$ event of two heads $= \{HH\}$

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{4}$$

[ex6] Two dice are thrown. Event A is "the sum is 7". Event B is "at least one die is a 6". Are A and B independent? [No]

Soln: Check if $P(A \cap B) = P(A)P(B)$

$$S = \{(1,1), (1,2), \dots, (1,6), (2,2), (2,3), \dots, (2,6), \dots\}$$

$$N(S) = 6 \times 6 = 36$$

$$A = \{(1,6), (2,5), (3,4), (6,1), (5,2), (4,3)\}$$

$$N(A) = 6$$

$$P(A) = \frac{N(A)}{N(S)} = \frac{6}{36} = \frac{1}{6}$$

$$B = \{\text{at least one of the two dice is a 6}\}$$

$$N(B^c) = 5 \times 5$$

$$P(B) = 1 - P(B^c)$$

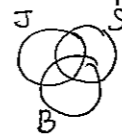
$$= 1 - \frac{5 \times 5}{36} = \frac{11}{36}$$

$$A \cap B = \{(1,6), (6,1)\}$$

$$P(A \cap B) = \frac{2}{36} \neq P(A)P(B) = \frac{1}{6} \times \frac{11}{36}$$

[ex7] The probability John will buy a product is $\frac{3}{5}$, that Bill will buy that product is $\frac{2}{3}$, and that Sue will buy that product is $\frac{1}{4}$. What's the probability that at least one of them will buy the product?

Soln: Whether John, Bill, or Sue buy the product are independent events. (They have no effect on each other).



Method 1 Probability at least one buys the product $= 1 - \text{Probability none of them buy it.}$

$$P(J \cup S \cup B) = 1 - P(J^c \cap S^c \cap B^c)$$

$$= 1 - P(J^c)P(S^c)P(B^c)$$

$$= 1 - \frac{2}{5} \times \frac{1}{3} \times \frac{3}{4} = \frac{9}{10}$$

$$P(J) = \frac{3}{5}$$

$$P(B) = \frac{2}{3}$$

$$P(S) = \frac{1}{4}$$

$$\text{Method 2: } P(J \cup S \cup B) = P(J) + P(S) + P(B) - P(J \cap S) - P(J \cap B) - P(S \cap B) + P(J \cap S \cap B)$$

$$= \frac{3}{5} + \frac{1}{4} + \frac{2}{3} - \frac{3}{5} \times \frac{1}{4} - \frac{3}{5} \times \frac{2}{3} - \frac{2}{3} \times \frac{1}{4} + \frac{3}{5} \times \frac{2}{3} \times \frac{1}{4} = \frac{9}{10}$$

[ex] In throwing two dice, event A is "the first die is 5". Event B is "the two dice sum to 2"

A and B are mutually exclusive $P(A \cap B) = 0$

because if the first die is 5, the second die (1~6) can never get a sum of 2

ex8 52-card deck

	Heart	Diamond	Spade	Club
	♥	♦	♠	♣
13	K Q J	10 9 8	7 6 5	4 3 2 A

what is the probability of drawing a spade or a king from a deck of 52 cards?

Soln: DNK = event of the king of spades $N(DNK)=1$
 D = event of a spade $N(D)=13$
 K = event of a king $N(K)=4$
 S = sample space of 52 cards $N(S)=52$

$$P(D \cup K) = P(D) + P(K) - P(DNK)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$

ex9 (Mutually Exclusive)

In a throw of two dice, what's the probability of getting a sum of 7 or 11?

Soln: $S = \{(1,1), (1,2), \dots, (2,6), \dots, (3,4)\}$ $N(S) = 6 \times 6$
 V = event of sum 7 = $\{(1,6), (2,5), (3,4)\}$ $N(V)=6$
 E = event of sum 11 = $\{(5,6), (6,5)\}$ $N(E)=2$

$$P(V \cup E) = P(V) + P(E) + P(V \cap E)$$

$$= \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$$

→ 0 mutually exclusive

ex10 You have 6 oranges, 3 apples. You randomly select 5 fruit to put in a basket. What is the probability of a basket with 3 oranges and 2 apples?

Soln: A = event of a basket with 3 oranges and 2 apples when you have 5 fruit total
 S = all possible combinations of 5 fruit order doesn't matter a_1, a_2, a_3, a_4, a_5

$N(A) = ?$ For each way you choose 3 from 6 oranges, there will be ways to choose 2 apples from 3 apples. Use the counting principle

$$N(A) = \binom{6}{3} \binom{3}{2}$$

$N(S) = \binom{9}{5}$ choose 5 out of 9 fruit, unordered.

$$P(A) = \frac{N(A)}{N(S)} = \frac{\binom{6}{3} \binom{3}{2}}{\binom{9}{5}} = \frac{10}{21} \text{ (use a calculator)}$$

ex The SAT Guessing Penalty

• Multiple Choice: 5 choice (A ~ E)

Each correct answer gets 1 point

(A) To account for lucky guesses, $\frac{1}{4}$ point is deducted for each incorrect answer. Why?

Soln: N_c = # correct answers that you did not guess on

$$N_g = N_{cg} + N_{wg}$$

Number guesses = Number correct guesses + Number of wrong guesses

• Assume that each time an answer is incorrect, it's because it's a wrong guess. $N_w = N_{wg}$

$$\text{Total score} = (N_c + N_{cg}) - \frac{1}{4} N_{wg}$$

ETS hopes that $N_{cg} \approx \frac{1}{4} N_{wg}$

because $N_{cg} \approx P(CG) N_g$ ($\frac{1}{5}$ th of the time, you're correct)

$$= \frac{1}{5} N_g \approx \frac{1}{5} \frac{5}{4} N_{wg} = \frac{1}{4} N_{wg}$$

$$N_{wg} \approx P(WG) N_g$$

$$= \frac{4}{5} N_g \text{ (4/5th of the time, you guess wrong)}$$

(B) How many answers should you eliminate to beat the odds? (Answer: Eliminate 1 or more choices)

Soln: $N_{cg} > \frac{1}{4} N_{wg}$. If you have n answers left over $P(CG) = \frac{1}{n}$ $P(WG) = \frac{n-1}{n}$

$$\frac{1}{4} < \frac{N_{cg}}{N_{wg}} \approx \frac{P(CG) N_g}{P(WG) N_g} = \frac{1/n}{(n-1)/n} = \frac{1}{n-1} \Rightarrow \begin{matrix} n < 5 \\ n \leq 4 \end{matrix}$$

II. Probability Appendix

- In Probability, you first learn about sample space, events, and counting $P(A) = \frac{N(A)}{N(S)}$
- Then, a random variable is introduced that maps an outcome to a number. For example, the outcome could be getting (1,6) in two rolls of dice. X could be the sum, $X((1,6)) = 7$. The random variable X maps the outcome (1,6) to a number 7.
- Then, the random variable is described by a probability density function, $f_X(x)$, representing how likely it is for X to be a particular value x . The significance is that a random variable is completely characterized by the pdf. The pdf summarizes the entire structure below. When you have a probability density function, the sample space and event space can be constructed from it.

$$(S, \mathcal{C}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{R}, P_X)$$

S : sample space, set of all possible outcomes in an experiment

$\mathcal{C}$: event space, class of all subsets of S , has 2^n elements if S has n elements

P : Probability function; assigns to each event in $\mathcal{C}$ a number in $[0,1]$

X : random variable, a function defined on S (e.g. sum of 2 rolls of dice, area of a disk)

$$\forall I \in \mathcal{R}, X^{-1}(I) = \{s \in S : X(s) \in I\} \in \mathcal{C}$$

For every interval of real numbers, its pre-image is an event

$\mathbb{R}$: the real number line

$\mathcal{R}$: "Borel sigma-algebra", set of all subsets of the real numbers

P_X : The distribution of X defined on $\mathcal{R}$.

$$P_X(I) = P(X \in I) = P(\{X^{-1}(I)\}) \quad \forall I \in \mathcal{R} \\ = P \circ X^{-1}(I)$$

F_X : the distribution function defined on $\mathbb{R}$

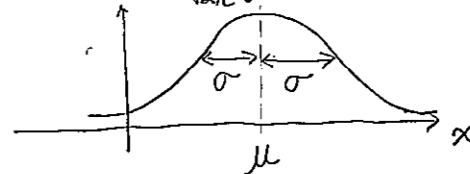
$$F_X(t) \equiv P(X \leq t) = P_X((-\infty, t])$$

Function	defined on
P	$\mathcal{C}$
P_X	$\mathcal{R}$
X	S
F_X	$\mathbb{R}$

- The probability density function can be approximated by a histogram.

A very important pdf is the Gaussian (or normal) probability density function. Many statistics follow this bell-shaped distribution.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



μ = mean, "average"
 σ = standard deviation

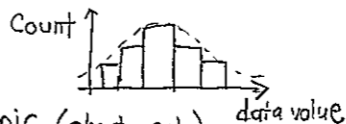
X is a random variable. It could be height, test score, ...

$f_X(x)$ represents how likely height $X = \text{certain value } x$ (5ft). It is most likely that the height of a randomly selected person in a population is the average height μ . If σ is small, almost everyone is $\approx \mu$ tall. If σ is large, there is a wider range of heights.

III. Statistics

- Data = set of numbers that refer to a variable (height, test grade, ...)

- data distribution as histogram



- a unimodal (1 peak) and symmetric (about peak) data distribution is summarized by its center (mean) and spread (standard deviation)

- Mean of $x_1, x_2, \dots, x_n$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{x_1 + x_2 + \dots + x_n}{n}$$

- Standard Deviation of $x_1, x_2, \dots, x_n$

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$$

- Median = middle value when data is arranged in ascending (or descending) order

⊗ The median of 1, 3, 5, 7, 9, 9 is $\frac{5+7}{2} = 6$

The median of 1, 3, 5 is 3

- Mode = the most frequent value in the data set

- Range = maximum value - minimum value

ex10 13 men traveled 18mph over the speed limit
8 women " 14mph "

What was the mean speed over the limit of all 21 drivers?

Soln: $\frac{13 \times 18 \text{ mph} + 8 \times 14 \text{ mph}}{21} = 16.48 \text{ mph}$

ex11

Data Value	Frequency
0	1
1	3
2	7
3	15
4	10
5	7
6	3
7	3

(A) median?

(B) mode?

(C) mean?

(D) Range?

(E) standard deviation?

(A) There are 49 data values. The median value

$$1 @ 3, \frac{3}{2} = 1.5 = 2$$

occurs at the $\frac{49}{2} = 25^{\text{th}}$ point, which is 3

(B) Mode = 3

(C) Mean = $\frac{0 \times 1 + 1 \times 3 + 2 \times 7 + \dots + 7 \times 3}{49} = 3.591 = \bar{x}$

(D) Range = $7 - 0 = 7$

(E) $\sigma = \sqrt{\frac{(0-\bar{x})^2 + 3(1-\bar{x})^2 + 7(2-\bar{x})^2 + 15(3-\bar{x})^2 + \dots + 3(7-\bar{x})^2}{48}}$
 ≈ 1.6

Easier way: Use a graphing calculator

STAT → Edit L3, L4

2nd LIST → MATH → mean (2nd 3 gives L3, L4) → 3.591

median(L3, L4) → 3

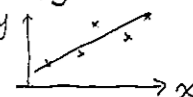
std Dev(L3, L4) → 1.60

Regression

Analyze the relationship between two variables.

Given data $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ you

can get a scatterplot



You can find the best-fit line and predict what y is, as a function of x. That's "Linear Regression".

Use Calculus: Find m, b such that

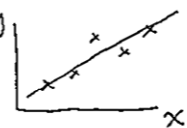
$$\sum_{i=1}^n (mx_i + b - y_i)^2 \text{ is a minimum}$$

As a function of m, this is convex

$$\frac{\partial}{\partial m} f(m, b) = 0$$

$$\frac{\partial}{\partial b} f(m, b) = 0$$

Linear Regression
 $\text{LinReg}(ax+b) (xlist, ylist, Y_1)$



$$y = ax + b$$

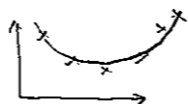
best-fit line

Exponential Regression
 ExpReg



$$y = ab^x$$

Quadratic Regression
 QuadReg



$$y = ax^2 + bx + c$$

ex12 Use exponential regression to estimate the 1965 population

Year	Population
1950	48,000
1960	72,000
1970	95,000
1980	123,000
1990	165,000
L5	L6

Soln:

- $\text{STAT} \rightarrow \text{Edit}$ L5, L6
- $\text{STAT} \rightarrow \text{CALC} \rightarrow \text{ExpReg}$
- $\text{ExpReg}(L5, L6, Y_1)$ This stores in Y_1
- $(xlist)(ylist)$ VARs $\rightarrow$ Y-VARS $\rightarrow$ Function $\rightarrow$ Y_1

$$\Rightarrow y = a * b^x$$

$$a = 1.799 \times 10^{-21}$$

$$b = 1.0305$$

$$Y_1(1965) = \boxed{79338.57}$$

• You can also view the plot & scatter plot

- 2nd STAT PLOT $\rightarrow$ Plot 1 $\rightarrow$ On
- $\rightarrow$ choose L5

XList: L5
 YList: L6

This plots the data in the table

- ZoomFit or choose window and trace

ex13 Jack recorded the amount of time he studied the night before each of 4 history quizzes and the score he got on each quiz. Use linear regression to estimate the score Jack would get if he studied for 20 minutes.

Score	Time (min)
86	45
70	15
90	40
78	35
L1	L2

Soln:

$\text{STAT} \rightarrow \text{Edit} \rightarrow L1, L2$

$\text{STAT} \rightarrow \text{CALC} \rightarrow \text{LinReg}(ax+b)$

$\text{LinReg}(ax+b) \text{ L2, L1, Y1}$
 $x \quad y \quad \text{VARs} \rightarrow \text{Y-VARS} \rightarrow \text{Function} \rightarrow Y_1$

$$\Rightarrow \text{LinReg}$$

$$y = ax + b \quad a = 0.607, b = 60.51$$

$$Y_1(20) = 72.65 \approx \boxed{73}$$

IV. Word Problems

ex14 (How many standard deviations above the mean?)

Test	Sonya's Score	Mean Score	Standard Deviation
V	80	81	4
W	90	84	2
X	96	86	5
Y	89	81	4
Z	88	85	3

on which of the five tests did Sonya score highest relative to the rest of the test takers

Soln: Eliminate Test V since Sonya scored below the mean.

Sonya's performance relative to others depends on how difficult the test was (the mean score and whether most people scored near the mean). Find the Test where Sonya was the most standard deviations above the mean

$$\textcircled{W} \quad \frac{90-84}{2} = 3 \text{ st. dev. above mean} \quad Y: \frac{89-81}{4} = 2$$

$$X: \frac{96-86}{5} = 2 \quad Z: \frac{88-85}{3} = 1$$

• Percent Increase / Decrease

"x increases by p%" means x increases to $x + \frac{p}{100}x$
 $= x(1 + \frac{p}{100})$

"x decreases by p%" means x decreases to $x(1 - \frac{p}{100})$

(See Lesson #7 Handout, Example 21)

• Investment

P = principal = amount invested into bank account
 Compound interest = interest earned on the principal
 and on the interest earned

[ex15] P = \$1000 invested at 12% annual interest ($r = 0.12$)
 compounded annually. After 3 years, what is the
 balance?

Soln: P
 After 1 year: $P + P \times r = (1+r)P$
 2 years: $(1+r)^2 P$
 n years: $(1+r)^n P$ $n=3$ $(1.12)^3 1000$
 $= \$1,405$

• Logic If $A \Rightarrow B$, then $B^c \Rightarrow A^c$ Contrapositive
 but $B \not\Rightarrow A$

⊗ If an object is a wheel, then it must be circular
 $A \Rightarrow B$

$B^c \Rightarrow A^c$: If an object is not circular,
 then it's not a wheel

$B \not\Rightarrow A$: If an object is circular,
 it might not be a wheel.
 (It could be a pizza)

• English to Algebra - be careful with units!

[ex16] Cost of salad is \$2 for first half-pound and
 75¢ for each additional half-pound. What's the
 total charge in cents for p pounds of salad,
 where p is a positive integer?

Soln: $p \geq 1$ lb., so the first 200¢ is charged.

$$200¢ + \frac{75¢}{\text{half-pound}} \times (p - \frac{1}{2}) \text{ pounds} \times 2 \frac{\text{half-pounds}}{\text{pound}}$$

$$= 200¢ + 75¢ \times (2p - 1) = \boxed{25(5 + 6p)}$$

• Averages

[ex17] The sales total for January was 50% greater than
 the average (arithmetic mean) of the monthly sales totals
 for February through December. The sales total for
 January was what fraction of the sales total for
 the year?

Soln: Let S_i = sales total for i^{th} month

$$\frac{1.5}{11} S_1 + S_1 = 1.5 \frac{S_2 + S_3 + \dots + S_{12} + S_1}{11} \quad \frac{S_1}{S_1 + \dots + S_{12}} = ?$$

$$\frac{12.5}{11} S_1 = \frac{1.5}{11} \sum_{i=1}^{12} S_i \Rightarrow \frac{S_1}{\sum_{i=1}^{12} S_i} = \frac{1.5}{12.5} = \boxed{\frac{3}{25}}$$

[ex18] (Distance = Rate $\times$ Time)

Kayla drove from Bayside to Chatham at a constant
 speed of 21 mph and then returned along the same
 route from Chatham to Bayside. If her average speed
 for the entire journey was 26.25 miles per hour, at
 what average speed, in miles per hour, did Kayla
 return from Chatham to Bayside?

Soln: Bayside $\xrightarrow[\text{?}]{\text{d}}$ Chatham
 $v_{BC} = 21 \text{ mph}$
 $v_{CB} = ?$

$$21 \text{ mph} \quad v_{BC} = \frac{d}{t_1} \Rightarrow t_1 = \frac{d}{21 \text{ mph}}$$

$$v_{CB} = \frac{d}{t_2} \Rightarrow t_2 = \frac{d}{v_{CB}}$$

$$26.25 \text{ mph} = \frac{2d}{\frac{d}{21 \text{ mph}} + \frac{d}{v_{CB}}} = \frac{2(21 \text{ mph})v_{CB}}{v_{CB} + 21 \text{ mph}}$$

$$26.25(v_{CB} + 21) = 42 v_{CB}$$

$$\boxed{v_{CB} = 35 \text{ mph}}$$